Control systems

Lecture-3: Time domain analysis

V. Sankaranarayanan
1 Review of Laplace Transform

2 Time domain analysis
   - First order system
   - Second order system
   - Steady state errors

3 Assignment
**Review of Complex Function**

**Complex Variable**

A complex variable has both real and imaginary part variable. It is generally represented as

\[ s = \sigma + j\omega \]

**Complex Function**

A complex function \( G(s) \) is a function of complex variable \( s \) which can be represented as

\[ G(s) = G_x + jG_y \]

where \( G_x \) and \( G_y \) is the real and imaginary component respectively.
**Complex Variable**

A complex variable has both real and imaginary part variable. It is generally represented as

\[ s = \sigma + j\omega \]

---

**Complex Function**

A complex function \( G(s) \) is a function of complex variable \( s \) which can be represented as

\[ G(s) = G_x + jG_y \]

where \( G_x \) and \( G_y \) is the real and imaginary component respectively.

- Magnitude of a complex variable is \( \sqrt{\sigma^2 + \omega^2} \)
**Complex Variable**

A complex variable has both real and imaginary part variable. It is generally represented as

\[ s = \sigma + j\omega \]

**Complex Function**

A complex function \( G(s) \) is a function of complex variable \( s \) which can be represented as

\[ G(s) = G_x + jG_y \]

where \( G_x \) and \( G_y \) is the real and imaginary component respectively.

- Magnitude of a complex variable is \( \sqrt{\sigma^2 + \omega^2} \)
- Angle with respect to real axis is \( \tan^{-1}(\omega/\sigma) \)
**Review of Complex Function**

**Complex Variable**

A complex variable has both real and imaginary part variable. It is generally represented as

\[ s = \sigma + j\omega \]

**Complex Function**

A complex function \( G(s) \) is a function of complex variable \( s \) which can be represented as

\[ G(s) = G_x + jG_y \]

where \( G_x \) and \( G_y \) is the real and imaginary component respectively.

- Magnitude of a complex variable is \( \sqrt{\sigma^2 + \omega^2} \)
- Angle with respect to real axis is \( \tan^{-1}(\omega/\sigma) \)
The Laplace transform can be defined as

\[ \mathcal{L}[f(t)] = F(s) = \int_{0-}^{\infty} f(t)e^{-st} \, dt \]
The Laplace transform can be defined as

$$\mathcal{L}[f(t)] = F(s) = \int_{0-}^{\infty} f(t)e^{-st}dt$$

- $f(t)$ is a function in time such that $f(t) = 0$ for $t < 0$
LAPLACE TRANSFORM-DEFINITION

**Definition**

The Laplace transform can be defined as

\[
\mathcal{L}[f(t)] = F(s) = \int_{0^-}^{\infty} f(t)e^{-st}dt
\]

- \(f(t)\) is a function in time such that \(f(t) = 0\) for \(t < 0\)
- If the integral exist then \(F(s)\) is called as *Laplace transform of \(f(t)\)*
**Definition**

The Laplace transform can be defined as

\[
\mathcal{L}[f(t)] = F(s) = \int_{0-}^{\infty} f(t)e^{-st} dt
\]

- \(f(t)\) is a function in time such that \(f(t) = 0\) for \(t < 0\)
- If the integral exist then \(F(s)\) is called as *Laplace transform of \(f(t)\)*
- The lower limit 0− allows us to integrate the function prior to origin even if there is a discontinuity at the origin.
The Laplace transform can be defined as

\[ \mathcal{L}[f(t)] = F(s) = \int_{0-}^{\infty} f(t)e^{-st} dt \]

- \( f(t) \) is a function in time such that \( f(t) = 0 \) for \( t < 0 \)
- If the integral exist then \( F(s) \) is called as Laplace transform of \( f(t) \)
- The lower limit 0— allows us to integrate the function prior to origin even if there is a discontinuity at the origin.
Impulse Function

Impulse function is a special limiting class of pulse function. It can be defined as

\[
g(t) = \lim_{a \to 0} \frac{A}{a} \quad \text{for} \quad 0 < t < a
\]

\[
= 0 \quad \text{for} \quad t < 0 \text{ and } t > a
\]

The height of function is \(A/a\) and duration is \(a\). Area under the function is \(A\).
Impulse function is a special limiting class of pulse function. It can be defined as

\[ g(t) = \lim_{a \to 0} \frac{A}{a} \]

for \(0 < t < a\)

\[ = 0 \]

for \(t < 0\) and \(t > a\)

The height of function is \(A/a\) and duration is \(a\). Area under the function is \(A\). Laplace Transform of impulse function is

\[
\mathcal{L}[g(t)] = \lim_{a \to 0} \left[ \frac{A}{as} (1 - e^{-as}) \right]
\]

\[ = \lim_{a \to 0} \frac{d}{da} A(1 - e^{-as}) \]

\[ = \frac{A}{s} \]

\[ = A \]
### Common Laplace Transform Pairs

<table>
<thead>
<tr>
<th>SL. No.</th>
<th>( f(t) )</th>
<th>( F(s) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Unit impulse ( \delta(t) )</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>Unit step ( u(t) )</td>
<td>( \frac{1}{s} )</td>
</tr>
<tr>
<td>3</td>
<td>( t )</td>
<td>( \frac{1}{s^2} )</td>
</tr>
<tr>
<td>4</td>
<td>( t^n ) ((n = 1, 2, 3...))</td>
<td>( \frac{n!}{s^{n+1}} )</td>
</tr>
<tr>
<td>5</td>
<td>( e^{-at} )</td>
<td>( \frac{1}{s + a} )</td>
</tr>
<tr>
<td>6</td>
<td>( \sin \omega t )</td>
<td>( \frac{\omega}{s^2 + \omega^2} )</td>
</tr>
<tr>
<td>7</td>
<td>( \cos \omega t )</td>
<td>( \frac{s}{s^2 + \omega^2} )</td>
</tr>
</tbody>
</table>
## Laplace Transform Theorems

<table>
<thead>
<tr>
<th>SL. No.</th>
<th>Theorem</th>
<th>Remark</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>( \mathcal{L}[k_1 f_1(t) + k_2 f_2(t)] = k_1 F_1(s) + k_2 F_2(s) )</td>
<td>Linearity</td>
</tr>
<tr>
<td>2</td>
<td>( \mathcal{L}[e^{-at} f(t)] = F(s + a) )</td>
<td>Frequency Shift theorem</td>
</tr>
<tr>
<td>3</td>
<td>( \mathcal{L}[f(t - T)] = e^{-sT} F(s) )</td>
<td>Time Shift theorem</td>
</tr>
<tr>
<td>4</td>
<td>( \mathcal{L}\left[\frac{d}{dt} f(t)\right] = sF(s) - f(0) )</td>
<td>Differentiation theorem</td>
</tr>
<tr>
<td>5</td>
<td>( \mathcal{L}\left[\int_0^t f(t) dt\right] = \frac{F(s)}{s} )</td>
<td>Integration theorem</td>
</tr>
<tr>
<td>6</td>
<td>( \mathcal{L}[tf(t)] = -\frac{dF(s)}{ds} )</td>
<td></td>
</tr>
<tr>
<td>7</td>
<td>( \mathcal{L}[f(\frac{t}{a})] = aF(sa) )</td>
<td>Scaling theorem</td>
</tr>
</tbody>
</table>
DEFINITION

If \( f(t) \) and \( df(t)/dt \) are Laplace transformable and if \( \lim_{t \to \infty} f(t) \) exist then

\[
\lim_{t \to \infty} f(t) = \lim_{s \to 0} sF(s)
\]
**Initial Value Theorem**

**Definition**

If $f(t)$ and $df(t)/dt$ are Laplace transformable and if $\lim_{s \to \infty} sF(s)$ exist then

$$f(0+) = \lim_{s \to \infty} sF(s)$$
Transfer Function

Representation

\[ G(s) = \frac{b_0 s^m + b_1 s^{m-1} + \ldots + b_{m-1} s + b_m}{a_0 s^n + a_1 s^{n-1} + \ldots + a_{n-1} s + a_n} \]
Transfer Function

**Representation**

\[
G(s) = \frac{b_0 s^m + b_1 s^{m-1} + \ldots + b_{m-1} s + b_m}{a_0 s^n + a_1 s^{n-1} + \ldots + a_{n-1} s + a_n}
\]

- \(m = n\) - Proper transfer function
## Transfer Function

<table>
<thead>
<tr>
<th>Representation</th>
</tr>
</thead>
<tbody>
<tr>
<td>[ G(s) = \frac{b_0 s^m + b_1 s^{m-1} + \ldots + b_{m-1} s + b_m}{a_0 s^n + a_1 s^{n-1} + \ldots + a_{n-1} s + a_n} ]</td>
</tr>
</tbody>
</table>

- \( m = n \) - Proper transfer function
- \( m < n \) - Strictly proper transfer function
Transfer function

Representation

\[ G(s) = \frac{b_0 s^m + b_1 s^{m-1} + \ldots + b_{m-1} s + b_m}{a_0 s^n + a_1 s^{n-1} + \ldots + a_{n-1} s + a_n} \]

- \( m = n \) - Proper transfer function
- \( m < n \) - Strictly proper transfer function
- \( m > n \) - Improper transfer function
Transfer Function

Representation

\[ G(s) = \frac{b_0 s^m + b_1 s^{m-1} + \ldots + b_{m-1} s + b_m}{a_0 s^n + a_1 s^{n-1} + \ldots + a_{n-1} s + a_n} \]

- \( m = n \) - Proper transfer function
- \( m < n \) - Strictly proper transfer function
- \( m > n \) - Improper transfer function
Convolution integral

**Convolution**

\[ G(s) = \frac{Y(s)}{U(s)} \]

\[ y(t) = \int_{0}^{t} g(t - \tau)u(\tau)d\tau \]
**Convolution integral**

\[ G(s) = \frac{Y(s)}{U(s)} \]

\[ y(t) = \int_0^t g(t - \tau)u(\tau) d\tau \]
Convolution integral

\[ G(s) = \frac{Y(s)}{U(s)} \]

\[ y(t) = \int_0^t g(t - \tau)u(\tau)d\tau \]
System definitions

Convolution integral

\[ y(t) = \int_{0}^{t} g(t - \tau)u(\tau)d\tau \]
**System definitions**

**Convolution integral**

\[ y(t) = \int_{0}^{t} g(t-\tau)u(\tau)d\tau \]

**Definition**

System is said to be time invariant if

\[ y(t) = \int_{0}^{t} g(t-\tau)u(\tau)d\tau \]

\[ = \int_{0}^{t} g(y)u(t-\tau)d\tau \]
System definitions

Convolution integral

\[ y(t) = \int_{0}^{t} g(t - \tau)u(\tau)d\tau \]

Definition

System is said to be time invariant if

\[ y(t) = \int_{0}^{t} g(t - \tau)u(\tau)d\tau \]

\[ = \int_{0}^{t} g(\tau)u(t - \tau)d\tau \]

Definition

System is said to be **CAUSAL** if \( \tau \leq t \)
Poles and Zeros

**Standard Form**

\[ G(s) = \frac{b_0 s^m + b_1 s^{m-1} + \ldots + b_{m-1} s + b_m}{a_0 s^n + a_1 s^{n-1} + \ldots + a_{n-1} s + a_n} \]
POLES AND ZEROS

Standard Form

\[ G(s) = \frac{b_0 s^m + b_1 s^{m-1} + \ldots + b_{m-1} s + b_m}{a_0 s^n + a_1 s^{n-1} + \ldots + a_{n-1} s + a_n} \]

Poles

Solution to the denominator polynomial called "poles"

\[ a_0 s^n + a_1 s^{n-1} + \ldots + a_{n-1} s + a_n = 0 \]
## Poles and Zeros

### Standard Form

\[ G(s) = \frac{b_0 s^m + b_1 s^{m-1} + \ldots + b_{m-1} s + b_m}{a_0 s^n + a_1 s^{n-1} + \ldots + a_{n-1} s + a_n} \]

### Poles

Solution to the denominator polynomial called "poles"

\[ a_0 s^n + a_1 s^{n-1} + \ldots + a_{n-1} s + a_n = 0 \]

### Zeros

Solution to the numerator polynomial called "zeros"

\[ b_0 s^m + b_1 s^{m-1} + \ldots + b_{m-1} s + b_m = 0 \]
POLES AND ZEROS

STANDARD FORM

\[ G(s) = \frac{b_0 s^m + b_1 s^{m-1} + \ldots + b_{m-1} s + b_m}{a_0 s^n + a_1 s^{n-1} + \ldots + a_{n-1} s + a_n} \]

POLES

Solution to the denominator polynomial called ”poles”

\[ a_0 s^n + a_1 s^{n-1} + \ldots + a_{n-1} s + a_n = 0 \]

ZEROS

Solution to the numerator polynomial called ”zeros”

\[ b_0 s^m + b_1 s^{m-1} + \ldots + b_{m-1} s + b_m = 0 \]

POLE-ZERO FORMAT

\[ G(s) \frac{(s + z_1)(s + z_2) + \ldots + (s + z_m)}{(s + p_1)(s + p_2) + \ldots + (s + p_n)} \]

V. Sankaranarayanan

Control system
**Type and Order of the System**

**Definition**

The highest power of the denominator polynomial is defined as order of the system.
# Type and Order of the System

## Definition
The highest power of the denominator polynomial is defined as **order** of the system.

## Example

### First order system

\[ G(s) = \frac{1}{s + 1} \]

### Second order system

\[ G(s) = \frac{s}{s^2 + 2s + 4} \]
TYPE AND ORDER OF THE SYSTEM

**Definition**
The highest power of the denominator polynomial is defined as order of the system.

**Example**
First order system

\[ G(s) = \frac{1}{s + 1} \]

Second order system

\[ G(s) = \frac{s}{s^2 + 2s + 4} \]

**Definition**
The number of poles located in the origin is defined as type of the system.
Type and Order of the System

**Definition**

The highest power of the denominator polynomial is defined as *order* of the system.

**Example**

First order system

\[ G(s) = \frac{1}{s + 1} \]

Second order system

\[ G(s) = \frac{s}{s^2 + 2s + 4} \]

**Definition**

The number of poles located in the origin is defined as *type* of the system.

**Example**

Type 1 system

\[ G(s) = \frac{1}{s(s + 2)} \]
**Impulse**

\[ u(t) = \lim_{t_0 \to 0} \frac{1}{t_0} \text{ for } 0 < t < t_0 \]

\[ = 0 \text{ for } t < 0 \text{ and } t > t_0 \]

\[ \int_{0}^{\infty} u(t)dt = 1 \]

\[ U(s) = 1 \]
Step

\[ u(t) = 1 \]

\[ U(s) = \frac{1}{s} \]
A ramp function $u(t) = t$ is shown in the figure. The Laplace transform of the ramp function is $U(s) = \frac{1}{s^2}$. This representation is useful in control system analysis for modeling and predicting system behavior under step changes in input.
Parabolic

\[ u(t) = t^2 \]

\[ U(s) = \frac{1}{s^3} \]
First order system - step input

First order system

\[ G(s) = \frac{1}{Ts + 1} \]

The output response for a step input \( Y(s) = \frac{U(s) \ast G(s)}{s} \)

\[ Y(s) = \frac{1}{s} \ast \frac{1}{Ts + 1} \]

Taking inverse Laplace transform

\[ y(t) = 1 - e^{-\frac{t}{T}} \]

at \( t = T \)

\[ y(T) = 1 - e^{-1} = 0.632 \]

at \( t = 2T \)

\[ y(2T) = 0 \]

\[ \frac{dy}{dt} \bigg|_{t=0} = T(e^{-\frac{t}{T}}-1) \]

\[ \frac{dy}{dt} \bigg|_{t=0} = T \]

V. Sankaranarayanan

Control system
**First order system - step input**

**First order system**

\[ G(s) = \frac{1}{Ts + 1} \]

The output response for a step input

\[ Y(s) = U(s) \ast G(s) \]

\[ Y(s) = \frac{1}{s} \ast \frac{1}{Ts + 1} \]

\[ Y(s) = \frac{1}{s} - \frac{T}{Ts + 1} \]

Taking inverse Laplace transform

\[ y(t) = 1 - e^{-t/T} \]
**First order system - step input**

**First order system**

\[ G(s) = \frac{1}{Ts + 1} \]

The output response for a step input

\[ Y(s) = U(s) * G(s) \]

\[ Y(s) = \frac{1}{s} * \frac{1}{Ts + 1} \]

\[ Y(s) = \frac{1}{s} - \frac{T}{Ts + 1} \]

Taking inverse Laplace transform

\[ y(t) = 1 - e^{-t/T} \]

- at \( t = T \)
  \[ y(T) = 1 - e^{-1} = 0.632 \]

- at \( t = 2T \)
  \[ y(2T) = 0.865 \]

\[ \frac{dy}{dt} \bigg|_{t=0} = \frac{1}{T} e^{-t/T} = \frac{1}{T} \]
First order system response - step input

\[ u(t) \] 

\[ y(t) \]

\[ T \]

\[ 2T \]

\[ 3T \]

\[ \frac{1}{T} \]

\[ 0.635 \]

\[ 0.865 \]

\[ 0.95 \]
First order system - ramp input

First order system

\[ G(s) = \frac{1}{Ts + 1} \]
First order system - ramp input

First order system

\[ G(s) = \frac{1}{Ts + 1} \]

The output response for a ramp input

\[ Y(s) = U(s) \ast G(s) \]

\[ Y(s) = \frac{1}{s^2} \ast \frac{1}{Ts + 1} \]

\[ Y(s) = \frac{1}{s^2} - \frac{T}{s} + \frac{T^2}{Ts + 1} \]

Taking inverse Laplace transform

\[ y(t) = t - T + Te^{-t/T} \]
First order system - ramp input

The output response for a ramp input

\[ Y(s) = U(s) \cdot G(s) \]
\[ Y(s) = \frac{1}{s^2} \cdot \frac{1}{Ts + 1} \]
\[ Y(s) = \frac{1}{s^2} - \frac{T}{s} + \frac{T^2}{Ts + 1} \]

Taking inverse Laplace transform

\[ y(t) = t - T + Te^{-t/T} \]
**FIRST ORDER SYSTEM - IMPULSE INPUT**

**FIRST ORDER SYSTEM**

\[ G(s) = \frac{1}{Ts + 1} \]

The output response for a impulse input

\[ Y(s) = U(s) \ast G(s) \]

\[ Y(s) = \frac{1}{Ts + 1} \]

Taking inverse Laplace transform

\[ y(t) = \frac{1}{T} e^{-t/T} \]
The output response for an impulse input can be expressed as:

\[ Y(s) = U(s) \ast G(s) \]

Setting \( G(s) \) to the given transfer function:

\[ G(s) = \frac{1}{Ts + 1} \]

The output response is:

\[ Y(s) = \frac{1}{Ts + 1} \]

Taking the inverse Laplace transform, the response becomes:

\[ y(t) = \frac{1}{T} e^{-t/T} \]
**SECOND ORDER SYSTEM**

**STANDARD REPRESENTATION**

\[ G(s) = \frac{\omega_n^2}{s^2 + 2\zeta \omega_n + \omega_n^2} \]

- \( \omega_n \) - Natural frequency
- \( \zeta \) - Damping factor

**POLE-ZERO FORM**

\[ \frac{\omega_n^2}{(s + \zeta \omega_n + j\omega_d)(s + \zeta \omega_n - j\omega_d)} \]

- \( \omega_d = \omega_n \sqrt{1 - \zeta^2} \) - Damped natural frequency
SECOND ORDER SYSTEM - STEP RESPONSE

- $0 < \zeta < 1$
- Step response

$$Y(s) = \frac{\omega_n^2}{(s + \zeta \omega_n + j \omega_d)(s + \zeta \omega_n - j \omega_d)s}$$

- Partial fraction

$$\frac{1}{s} = \frac{s + \zeta \omega_n}{(s + \zeta \omega_n)^2 + \omega_d^2} - \frac{\zeta \omega_n}{(s + \zeta \omega_n)^2 + \omega_d^2}$$

- Taking inverse Laplace transform

$$y(t) = 1 - e^{-\zeta \omega_n t} \left( \cos \omega_d t + \frac{\zeta}{\sqrt{1 - \zeta^2}} \sin \omega_d t \right)$$

- further

$$y(t) = 1 - \frac{e^{-\zeta \omega_n t}}{\sqrt{1 - \zeta^2}} \sin \left( \omega_d t + \tan^{-1} \frac{\sqrt{1 - \zeta^2}}{\zeta} \right)$$

V. Sankaranarayanan Control system
SECOND ORDER SYSTEM - STEP RESPONSE

- $\zeta = 1$
- Step response

$$Y(s) = \frac{\omega_n^2}{(s + \omega_n)^2 s}$$

- Taking inverse Laplace transform

$$y(t) = 1 - e^{-\omega_n t}(1 + \omega_n t)$$
Response

Step Response

Change in Response with $\zeta$

Time (seconds)

Amplitude

V. Sankaranarayanan

Control system
Review of Laplace Transform

Time domain analysis

Assignment

First order system

Second order system

Steady state errors

System Response With Pole Position

V. Sankaranarayanan

Control system
BASIC DEFINITIONS

- Delay time \( t_d \) - The delay time is the time required for the response to reach half of the final value the very first time
- Rise time \( t_s \) - The rise time is the time required for the response to rise from 0 – 100% of its final value
- Peak time \( t_p \) - The peak time is the time required for the response to reach the first peak of the overshoot
- Maximum overshoot \( M_p \)
  \[
  \frac{y(t_p) - c(\infty)}{c(\infty)} \times 100% 
  \]
- Settling time \( t_s \) - Settling time is the time required for the response curve to reach and stay within the range about the final value
SECOND ORDER STEP RESPONSE

\[ y(t) \]

- \( t_d \)
- \( t_r \)
- \( t_p \)
- \( t_s \)
Time domain specifications

Rise time

\[ t_r = \frac{1}{\omega_d} \tan^{-1} \left( \frac{\omega_d}{-\zeta \omega_n} \right) \]
### Time Domain Specifications

#### Rise Time

\[
tr = \frac{1}{\omega_d} \tan^{-1} \left( \frac{\omega_d}{-\zeta \omega_n} \right)
\]

#### Peak Time

\[
t_p = \frac{\pi}{\omega_d}
\]
TIME DOMAIN SPECIFICATIONS

RISE TIME

\[ t_r = \frac{1}{\omega_d} \tan^{-1} \left( \frac{\omega_d}{-\zeta \omega_n} \right) \]

PEAK TIME

\[ t_p = \frac{\pi}{\omega_d} \]

MAXIMUM OVERSHEEPT

\[ M_p = e^{\left(-\frac{\zeta}{\sqrt{1-\zeta^2}}\right)\pi} \times 100\% \]
## Time Domain Specifications

### Rise Time

\[
tr = \frac{1}{\omega_d} \tan^{-1}\left(\frac{\omega_d}{-\zeta \omega_n}\right)
\]

### Peak Time

\[
 tp = \frac{\pi}{\omega_d}
\]

### Maximum Overshoot

\[
 Mp = e^{-\left(\frac{\zeta}{\sqrt{1-\zeta^2}}\right)\pi} \times 100\%
\]

### Settling Time

\[
 ts = \frac{4}{\zeta \omega_n}
\]
Consider a generalized transfer function

\[ G(s) = k \frac{\sum_{i=1}^{m} (s - z_i)}{\sum_{i=1}^{n} (s - p_i)} \]

\[ = \frac{c_1}{s - p_1} + \frac{c_2}{s - p_2} + \ldots + \frac{c_n}{s - p_n} \text{ if all poles are distinct} \]

where \( c_1, c_2, \ldots c_n \) are the residue of \( G(s) \) at \( p_i \). So the inclusion of zeros only affecting the values of \( c_i \).

For poles with multiplicity, partial fraction can be done in similar way with some change in computation of \( c \).
SYSTEM RESPONSE WITH ZEROS

Consider the step response of the following systems

**No Zero**

\[ G(s) = \frac{1}{s + 2} \quad U(s) = \frac{1}{s} \]

\[ y(t) = \frac{1}{2} (1 - e^{-2t}) \]
WITH ZERO

\[ G(s) = \frac{s + 1}{s + 2} \]
\[ U(s) = \frac{1}{s} \]
\[ Y(s) = \frac{s + 1}{s(s + 2)} \]
\[ = \frac{0.5}{s} + \frac{0.5}{s + 2} \]
\[ y(t) = 0.5 + 0.5e^{-2t} \]
Response with Zero

Let $Y(s)$ be the response of a system $G(s)$, with unity in the numerator. The response of $(s + a)G(s)$ is

$$(s + a)Y(s) = sY(s) + aY(s)$$
**Response with Zero**

Let \( Y(s) \) be the response of a system \( G(s) \), with unity in the numerator. The response of \( (s + a)G(s) \) is

\[
(s + a)Y(s) = sY(s) + aY(s)
\]

- First part is the derivative of the original response.
- Second part is a scaled version of the original response.
System Response with Zeros

Response with Zero

Let $Y(s)$ be the response of a system $G(s)$, with unity in the numerator. The response of $(s + a)G(s)$ is

$$(s + a)Y(s) = sY(s) + aY(s)$$

- First part is the derivative of the original response.
- Second part is a scaled version of the original response.
- For large value of $a$ a scaled version of original response.
- Smaller value of $a$ derivative term will contribute to response.
Response with Zero

Let $Y(s)$ be the response of a system $G(s)$, with unity in the numerator. The response of $(s + a)G(s)$ is

$$(s + a)Y(s) = sY(s) + aY(s)$$

- First part is the derivative of the original response.
- Second part is a scaled version of the original response.
- For large value of $a$ a scaled version of original response.
- Smaller value of $a$ derivative term will contribute to response.
- For a second order system initial derivative of $Y(s)$ is positive. So a increase in overshoot is expected.
Let $Y(s)$ be the response of a system $G(s)$, with unity in the numerator. The response of $(s + a)G(s)$ is

$$(s + a)Y(s) = sY(s) + aY(s)$$

- First part is the derivative of the original response.
- Second part is a scaled version of the original response.
- For large value of $a$ a scaled version of original response.
- Smaller value of $a$ derivative term will contribute to response.
- For a second order system initial derivative of $Y(s)$ is positive. So a increase in overshoot is expected.
**Example**

Consider the following systems

\[ T_1(s) = \frac{4}{s^2 + 1.2s + 4} \quad T_2(s) = \frac{4(s + 1)}{s^2 + 1.2s + 4} \quad T_3(s) = \frac{4(s + 15)}{15(s^2 + 1.2s + 4)} \]

The response of systems are

\[ y_1(t) = 1 - 0.95393e^{-0.6t} \sin(1.9078t + 72.54^\circ) \]
\[ y_2(t) = 1 - e^{-0.6t}(\cos 1.9078t - 1.7820 \sin 1.9078t) \]
\[ y_2(t) = 1 - e^{-0.6t}(\cos 1.9078t + 0.1747 \sin 1.9078t) \]
Consider a system

\[ G(s) = \frac{1}{(s + a)(s^2 + bs + c)} \]

Impulse response of the system is

\[ y(t) = Ae^{-at} + \text{2nd order response} \]

- If \( a \) is very far from imaginary axis the exponential response will die very fast.
Consider a system

\[ G(s) = \frac{1}{(s + a)(s^2 + bs + c)} \]

Impulse response of the system is

\[ y(t) = Ae^{-at} + \text{2nd order response} \]

- If \( a \) is very far from imaginary axis the exponential response will die very fast.
- So here the 2nd order poles are dominant and \( a \) is a insignificant pole
Consider a system

\[ G(s) = \frac{1}{(s + a)(s^2 + bs + c)} \]

Impulse response of the system is

\[ y(t) = Ae^{-at} + \text{2nd order response} \]

- If \( a \) is very far from imaginary axis the exponential response will die very fast.
- So here the 2nd order poles are dominant and \( a \) is a insignificant pole
- If the magnitude of a real part of a pole is more than 5 to 10 times the real part of dominant pole that the pole is considered as insignificant
Consider a system

\[ G(s) = \frac{1}{(s + a)(s^2 + bs + c)} \]

Impulse response of the system is

\[ y(t) = Ae^{-at} + \text{2nd order response} \]

- If \( a \) is very far from imaginary axis the exponential response will die very fast.
- So here the 2nd order poles are dominant and \( a \) is a insignificant pole
- If the magnitude of a real part of a pole is more than 5 to 10 times the real part of dominant pole that the pole is considered as insignificant
Example

Consider the following systems

\[ T_1(s) = \frac{4}{s^2 + 1.2s + 4} \]
\[ T_2(s) = \frac{4}{(s + 1)(s^2 + 1.2s + 4)} \]
\[ T_3(s) = \frac{28}{(s + 7)(s^2 + 1.2s + 4)} \]

Poles = \(-0.6 \pm 1.9078i\)  
Poles = \(-1, -0.6 \pm 1.9078i\)  
Poles = \(-7, -0.6 \pm 1.9078\)

\[ y_1(t) = 1 - 0.95393e^{-0.6t} \sin (1.9078t + 72.54^\circ) \]
\[ y_2(t) = 1 - 1.0526e^{-t} - e^{-0.6t} (0.5351 \sin 1.9078t - 0.0526 \cos 1.9078) \]
\[ y_3(t) = 1 - 0.085e^{-7t} - e^{-0.6t} (0.9103 \cos 1.9078t + 0.6153 \sin 1.9078t) \]
Consider the following systems

\[ T_1(s) = \frac{4}{s^2 + 1.2s + 4} \quad T_2(s) = \frac{4}{(s^2 + 1.6s + 1)(s^2 + 1.2s + 4)} \quad T_3(s) = \frac{400}{(s^2 + 16s + 100)(s^2 + 1.2s + 4)} \]

Poles = \(-0.6 \pm 1.9078i\) \quad Poles = \(-0.8 \pm 0.6, -0.6 \pm 1.9078i\) \quad Poles = \(-8 \pm 6, -0.6 \pm 1.9078\)

\[
y_1(t) = 1 - 0.95393e^{-0.6t} \sin(1.9078t + 72.54^\circ)
\]
\[
y_2(t) = 1 - e^{-0.8t}(1.51 \sin 0.6t + 1.314 \cos 0.6t) + e^{-0.6t}(0.023 \sin 1.9078t + 0.314 \cos 1.9078t)
\]
\[
y_3(t) = 1 + e^{-8t}(0.028 \sin 6t - 0.067 \cos 6t) - e^{-0.6t}(0.9328 \cos 1.9078t + 0.6633 \sin 1.9078t)
\]
Open-loop

\[ G(s) = \frac{Y(s)}{U(s)} \]
**CLOSED-LOOP SYSTEM**

\[ G(s) = \frac{Y(s)}{U(s)} \]
Closed-loop system

\[ G(s) = \frac{Y(s)}{U(s)} \]

\[ \frac{C(s)}{R(s)} = \frac{G(s)}{1 + G(s)} \]
**CLOSED-LOOP SYSTEM**

\[ G(s) = \frac{Y(s)}{U(s)} \]

\[ \frac{C(s)}{R(s)} = \frac{G(s)}{1 + G(s)} \]

\[ E(s) = \frac{R(s)}{1 + G(s)} \]
Review of Laplace Transform
Time domain analysis
Assignment
First order system
Second order system
Steady state errors

CLOSED-LOOP WITH CONTROLLERS

[Diagram showing a control system with a block diagram of a closed-loop system. The diagram includes symbols for R(s), E(s), Gc(s), G(s), and C(s). The loop transfer function is indicated by a line connecting Gc(s) and G(s).]
The main of any closed loop control system is to make the error between the desired output and the actual output is zero

\[ E(s) = \frac{R(s)}{1 + G(s)} \]
The main of any closed loop control system is to make the error between the desired output and the actual output is zero

\[ E(s) = \frac{R(s)}{1 + G(s)} \]

Taking inverse Laplace transform

\[ e(t) = ? \]
The main of any closed loop control system is to make the error between the desired output and the actual output is zero

\[ E(s) = \frac{R(s)}{1 + G(s)} \]

Taking inverse Laplace transform

\[ e(t) =? \]

when we apply Final value theorem

\[ e_{ss} = \lim_{t \to \infty} e(t) = \lim_{s \to 0} sE(s) \]
The main of any closed loop control system is to make the error between the desired output and the actual output is zero

\[ E(s) = \frac{R(s)}{1 + G(s)} \]

Taking inverse Laplace transform

\[ e(t) = ? \]

when we apply **Final value theorem**

\[ e_{ss} = \lim_{t \to \infty} e(t) = \lim_{s \to 0} sE(s) \]

Steady state error

\[ e_{ss} = \lim_{s \to 0} \frac{sR(s)}{1 + G(s)} \]
**Type 0 System**

\[ G(s) = \frac{K(T_{z1}s + 1)(T_{z2}s + 1) + \ldots (T_{zm}s + 1)}{(T_{p1}s + 1)(T_{p2}s + 1) + \ldots + (T_{pn}s + 1)} \]
**Type 0 System**

\[ G(s) = \frac{K(T_{z1}s + 1)(T_{z2}s + 1) + \ldots (T_{zm}s + 1)}{(T_{p1}s + 1)(T_{p2}s + 1) + \ldots + (T_{pn}s + 1)} \]

**Step Input**

\[ e_{ss} = \lim_{s \to 0} sE(s) = \lim_{s \to 0} \frac{sR(s)}{s(1 + G(s))} = \lim_{s \to 0} \frac{1}{1 + G(s)} = \frac{1}{1 + K} \]
Type 0 System

\[ G(s) = \frac{K(T_{Z1}s + 1)(T_{Z2}s + 1) + \ldots (T_{Zm}s + 1)}{(T_{P1}s + 1)(T_{P2}s + 1) + \ldots + (T_{Pn}s + 1)} \]

Step Input

\[ e_{ss} = \lim_{s \to 0} sE(s) = \lim_{s \to 0} sR(s) = \lim_{s \to 0} \frac{s}{s(1 + G(s))} = \lim_{s \to 0} \frac{1}{1 + G(s)} = \frac{1}{1 + K} \]

Ramp Input

\[ e_{ss} = \lim_{s \to 0} sE(s) = \lim_{s \to 0} sR(s) = \lim_{s \to 0} \frac{s}{s^2(1 + G(s))} = \lim_{s \to 0} \frac{1}{s(1 + G(s))} = \infty \]
Type 1 System

\[ G(s) = \frac{K(T_{z1}s + 1)(T_{z2}s + 1) + \cdots (T_{zm}s + 1)}{s(T_{p1}s + 1)(T_{p2}s + 1) + \cdots + (T_{pn}s + 1)} \]
**Type 1 System**

\[ G(s) = \frac{K(T_{z1}s + 1)(T_{z2}s + 1) + \ldots (T_{zm}s + 1)}{s(T_{p1}s + 1)(T_{p2}s + 1) + \ldots + (T_{pn}s + 1)} \]

**Step Input**

\[ e_{ss} = \lim_{s \to 0} sE(s) = \lim_{s \to 0} \frac{sR(s)}{1 + G(s)} = \lim_{s \to 0} \frac{s}{s(1 + G(s))} = \lim_{s \to 0} \frac{1}{1 + G(s)} = \frac{1}{1 + \infty} = 0 \]
**Type 1 system**

\[ G(s) = \frac{K(T_{z1}s + 1)(T_{z2}s + 1) + \ldots (T_{zm}s + 1)}{s(T_{p1}s + 1)(T_{p2}s + 1) + \ldots + (T_{pn}s + 1)} \]

**Step input**

\[ e_{ss} = \lim_{s \to 0} sE(s) = \lim_{s \to 0} \frac{sR(s)}{1 + G(s)} = \lim_{s \to 0} \frac{s}{s(1 + G(s))} = \lim_{s \to 0} \frac{1}{1 + G(s)} = \frac{1}{1 + \infty} = 0 \]

**Ramp input**

\[ e_{ss} = \lim_{s \to 0} sE(s) = \lim_{s \to 0} \frac{sR(s)}{1 + G(s)} = \lim_{s \to 0} \frac{s}{s^2(1 + G(s))} = \lim_{s \to 0} \frac{1}{s + sG(s)} = \frac{1}{K} \]
**Type 1 System**

\[ G(s) = \frac{K(T_{z1}s + 1)(T_{z2}s + 1) + \ldots (T_{zm}s + 1)}{s(T_{p1}s + 1)(T_{p2}s + 1) + \ldots + (T_{pn}s + 1)} \]

**Step Input**

\[ e_{ss} = \lim_{s \to 0} sE(s) = \lim_{s \to 0} \frac{sR(s)}{1 + G(s)} = \lim_{s \to 0} \frac{s}{s(1 + G(s))} = \lim_{s \to 0} \frac{1}{1 + G(s)} = \frac{1}{1 + \infty} = 0 \]

**Ramp Input**

\[ e_{ss} = \lim_{s \to 0} sE(s) = \lim_{s \to 0} \frac{sR(s)}{1 + G(s)} = \lim_{s \to 0} \frac{s}{s^2(1 + G(s))} = \lim_{s \to 0} \frac{1}{s + sG(s)} = \frac{1}{K} \]

**Parabolic Input**

\[ e_{ss} = \lim_{s \to 0} sE(s) = \lim_{s \to 0} \frac{sR(s)}{1 + G(s)} = \lim_{s \to 0} \frac{s}{s^3(1 + G(s))} = \lim_{s \to 0} \frac{1}{s^2 + s^2G(s)} = \frac{1}{0} = \infty \]
Type 2 system

\[ G(s) = \frac{K(T_{z1}s + 1)(T_{z2}s + 1) + \ldots (T_{zm}s + 1)}{s^2(T_{p1}s + 1)(T_{p2}s + 1) + \ldots + (T_{pn}s + 1)} \]
**Type 2 System**

\[ G(s) = \frac{K(T_{z1}s + 1)(T_{z2}s + 1) + \ldots (T_{zm}s + 1)}{s^2(T_{p1}s + 1)(T_{p2}s + 1) + \ldots + (T_{pn}s + 1)} \]

**Step Input**

\[ e_{ss} = \lim_{s \to 0} sE(s) = \lim_{s \to 0} \frac{sR(s)}{1 + G(s)} = \lim_{s \to 0} \frac{s}{s(1 + G(s))} = \lim_{s \to 0} \frac{1}{1 + G(s)} = \frac{1}{1 + \infty} = 0 \]
### Type 2 System

**Type 2 System**

\[ G(s) = \frac{K(Tz_1s + 1)(Tz_2s + 1) + \ldots (Tz_ms + 1)}{s^2(Tp_1s + 1)(Tp_2s + 1) + \ldots + (Tp_ns + 1)} \]

### Step Input

\[ e_{ss} = \lim_{s \to 0} sE(s) = \lim_{s \to 0} \frac{sR(s)}{1 + G(s)} = \lim_{s \to 0} \frac{s}{s(1 + G(s))} = \lim_{s \to 0} \frac{1}{1 + G(s)} = \frac{1}{1 + \infty} = 0 \]

### Ramp Input

\[ e_{ss} = \lim_{s \to 0} sE(s) = \lim_{s \to 0} \frac{sR(s)}{1 + G(s)} = \lim_{s \to 0} \frac{s}{s^2(1 + G(s))} = \lim_{s \to 0} \frac{1}{s + sG(s)} = \frac{1}{\infty} = 0 \]
TYPE 2 SYSTEM

\[ G(s) = \frac{K(T_{z1}s + 1)(T_{z2}s + 1) + \ldots (T_{zm}s + 1)}{s^2(T_{p1}s + 1)(T_{p2}s + 1) + \ldots + (T_{pn}s + 1)} \]

STEP INPUT

\[ e_{ss} = \lim_{s \to 0} sE(s) = \lim_{s \to 0} \frac{sR(s)}{1 + G(s)} = \lim_{s \to 0} \frac{s}{s(1 + G(s))} = \lim_{s \to 0} \frac{1}{1 + G(s)} = \frac{1}{1 + \infty} = 0 \]

RAMP INPUT

\[ e_{ss} = \lim_{s \to 0} sE(s) = \lim_{s \to 0} \frac{sR(s)}{1 + G(s)} = \lim_{s \to 0} \frac{s}{s^2(1 + G(s))} = \lim_{s \to 0} \frac{1}{s + sG(s)} = \frac{1}{\infty} = 0 \]

PARABOLIC INPUT

\[ e_{ss} = \lim_{s \to 0} sE(s) = \lim_{s \to 0} \frac{sR(s)}{1 + G(s)} = \lim_{s \to 0} \frac{s}{s^3(1 + G(s))} = \lim_{s \to 0} \frac{1}{s^2 + s^2G(s)} = \frac{1}{K} \]
### Steady State Error Constants

<table>
<thead>
<tr>
<th>Type</th>
<th>Step Input</th>
<th>Ramp Input</th>
<th>Parabolic input</th>
</tr>
</thead>
<tbody>
<tr>
<td>Type 0 system</td>
<td>( \frac{1}{1+K} )</td>
<td>( \infty )</td>
<td>( \infty )</td>
</tr>
<tr>
<td>Type 1 system</td>
<td>0</td>
<td>( \frac{1}{K} )</td>
<td>( \infty )</td>
</tr>
<tr>
<td>Type 2 system</td>
<td>0</td>
<td>0</td>
<td>( \frac{1}{K} )</td>
</tr>
</tbody>
</table>
1. Show that \( \mathcal{L}[-tf(t)] = \frac{dF(s)}{ds} \), where \( F(s) = \mathcal{L}[f(t)] \)
2. Apply final value theorem to the system transfer function

\[ G(s) = \frac{160s^3 + 188s^2 + 29s + 1}{260s^3 + 268s^2 + 46s + 2} \]
3. Find the steady state error of the unity feedback system shown below to a step and parabolic inputs.

\[ G(s) = \frac{10}{s^2(s + 1)} \]
4. Consider the system shown in the figure. If $K_0 = 0$, determine the damping factor and natural frequency of the system. What is the steady-state error resulting from unit ramp input?
5. Consider the system shown in the figure. Determine the feedback constant $K_0$, which will increase damping factor of the system to 0.6. What is the steady-state error resulting from unit ramp input with this setting of the feedback constant.
6. A certain system is described by the differential equation
\[ \ddot{y} + b\dot{y} + 4 = r \]

Determine the value of \( b \) such that \( M_p \) to be as small as possible but not greater than 15%.
7. Consider the system shown in the figure. Determine the system type.
8. Consider the system shown in the figure. To yield 0.1% error in the steady state to step input find the value of $K$. 

\[
\frac{(s + 1)}{s^2(s + 2)}
\]
9. A unity feedback system is characterized by the open loop transfer function.

\[ G(s) = \frac{1}{s(0.5s + 1)(0.2s + 1)} \]

Determine the steady state state errors for the unit step, unit ramp and unit acceleration inputs. Also determine the damping ratio and natural frequency of the dominant roots.
10. A speed control system of an armature-controlled dc motor as shown in the figure uses the back emf voltage of the motor as a feedback signal. (i) Calculate the steady-state error of this system to a step input command setting the speed to a new level. Assume that $R_a=L_a=J=b=1$, the motor constant is $K_m=1$, and $K_b=1$. (ii) Select a feedback gain for the back emf signal to yield a step response with an overshoot of 15%.