Modern Control systems

Lecture-3. Solution of State Equations

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Outline

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1. Solution of Differential Equation
   - Solution of Scalar D.E.s
   - Solution of Vector D.E.s

2. State Transition Matrix
   - Properties of State Transition Matrix

3. Computational Methods of Matrix Exponential
   - Laplace Transformation Approach
   - Diagonal Transformation
   - Cayley-Hamilton Theorem Approach

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Solution to Homogeneous State Equations

Solution to Scalar D.E.s

- Let us consider the scalar differential equation, $\dot{x} = ax$, where

$$x(t) = b_0 + b_1 t + b_2 t^2 + \cdots + b_k t^k + \cdots$$

- On substituting $x(t)$ in our scalar differential Equation, we get..

$$b_1 + 2b_2 t + 3b_3 t^2 + \cdots + k b_k t^{k-1} + \cdots = a(b_0 + b_1 t + b_2 t^2 + \cdots + b_k t^k + \cdots)$$

- Equating the coefficients of equal powers of 't'

$$\begin{aligned} b_1 &= ab_0 \\ b_2 &= \frac{1}{2}ab_1 = \frac{1}{2}a^2 b_0 \\ b_3 &= \frac{1}{3}ab_2 = \frac{1}{2*3}a^3 b_0 \\ &\vdots \\ b_k &= \frac{1}{k!}a^k b_0 \end{aligned}$$

Solution of Homogeneous State Equations

Continuation of Solution of scalar D.E.s...

- The value of b_0 can be determined by substituting $t = 0$ in $x(t) = b_0 + b_1 t + b_2 t^2 + \cdots + b_k t^k + \cdots$

$$x(0) = b_0$$

- Hence the solution x(t) can be written as,

$$x(t) = (1 + at + \frac{1}{2!}a^2t^2 + \frac{1}{3!}a^3t^3 + \cdots + \frac{1}{k!}a^kt^k + \cdots)b_0$$

$$= (1 + at + \frac{1}{2!}a^2t^2 + \frac{1}{3!}a^3t^3 + \cdots + \frac{1}{k!}a^kt^k + \cdots)x(0)$$

$$= e^{at}x(0)$$

Outline

Solution of Homogeneous State Equations

Solution of Vector-matrix D.E.s

- Let us consider the vector differential equation,

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x},$$

where, $\mathbf{x} \in \mathbb{R}^n \rightarrow$ n-vector

$\mathbf{A} \in \mathbb{R}^{n*n} \rightarrow n * n$ constant matrix

Solution of Homogeneous State Equations

Solution of Vector-matrix D.E.s

- Let,

$$\begin{aligned}\dot{x}_1(t) &= a_{11}x_1(t) + a_{12}x_2(t) + \cdots + a_{1n}x_n(t) \\ \dot{x}_2(t) &= a_{21}x_1(t) + a_{22}x_2(t) + \cdots + a_{2n}x_n(t) \\ &\vdots \\ &\vdots \\ \dot{x}_n(t) &= a_{n1}x_1(t) + a_{n2}x_2(t) + \cdots + a_{nn}x_n(t)\end{aligned} \quad (1)$$

- This can be written in Matrix form as,

$$\begin{bmatrix}\dot{x}_1(t) \\ \vdots \\ \dot{x}_n(t)\end{bmatrix} = \begin{bmatrix}a_{11} & \cdots & a_{n1} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn}\end{bmatrix}\begin{bmatrix}x_1(t) \\ \vdots \\ x_n(t)\end{bmatrix}$$

Solution of Homogeneous State Equations

Solution of Vector D.E.s

- Let,

$$x_1(t) = a_0 + a_1 t + a_2 t^2 + \cdots + a_n t^n \quad (2)$$
$$x_2(t) = b_0 + b_1 t + b_2 t^2 + \cdots + b_n t^n \quad (3)$$
$$x_3(t) = c_0 + c_1 t + c_2 t^2 + \cdots + c_n t^n \quad (4)$$
$$\vdots$$

- Consider the equation,
  $x_1(t) = a_0 + a_1 t + a_2 t^2 + \cdots + a_n t^n$
- Differentiating with respect to t, we get,
  $\dot{x_1}(t) = a_1 + 2a_2 t + 3a_3 t^2 + \cdots + n a_n t^{n-1}$

  Substituting this in equation (1) , we get,
  $a_1 + 2a_2 t + 3a_3 t^2 + \cdots + n a_n t^{n-1} =$
  $a_{11}(a_0 + a_1 t + a_2 t^2 + \cdots + a_n t^n) + a_{12}(b_0 + b_1 t + b_2 t^2 + \cdots + b_n t^n) +$
  $a_{13}(c_0 + c_1 t + c_2 t^2 + \cdots + c_n t^n) + ...$

Solution of Homogeneous State Equations

Solution of Vector-matrix D.E.s

- Equating coefficients of equal powers of 't',

$$\begin{aligned} a_1 &= a_{11}a_0 + a_{12}b_0 + a_{13}c_0 + \ldots \\ b_1 &= a_{21}a_0 + a_{22}b_0 + a_{23}c_0 + \ldots \\ c_1 &= a_{31}a_0 + a_{32}b_0 + a_{33}c_0 + \ldots \\ &\vdots \end{aligned}$$

- Similarly,

$$a_2 = a_{11}a_1 + a_{12}b_1 + a_{13}c_1 + ..$$

$$= a_{11}(a_{11}a_0 + a_{12}b_0 + a_{13}c_0 + \ldots) + a_{12}(a_{21}a_0 + a_{22}b_0 + a_{23}c_0 + \ldots) + \cdots$$

Solution of Homogeneous State Equations

Solution of Vector-matrix D.E.s

- Substituting $t = 0$ in equations (2), (3), $\cdots$ we get

$$\begin{aligned} x_1(0) &= a_0 \\ x_2(0) &= b_0 \\ x_3(0) &= c_0 \\ &\vdots \end{aligned}$$

Solution of Homogeneous State Equations

Solution of Vector-matrix D.E.s

- Summing up all the results obtained so far, we get

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \end{bmatrix} = \begin{bmatrix} 1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 1 \end{bmatrix} \begin{bmatrix} a_0 \\ b_0 \\ c_0 \\ \vdots \end{bmatrix} + \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} a_0 \\ b_0 \\ c_0 \\ \vdots \end{bmatrix} t + \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix}^2 \begin{bmatrix} a_0 \\ b_0 \\ c_0 \\ \vdots \end{bmatrix} t^2 + \cdots$$

Solution of Homogeneous State Equations

Solution of Vector-matrix D.E.s

- Replacing $a_0, b_0, \cdots with x_1(0), x_2(0), \cdots$ we get,

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \end{bmatrix} = \begin{bmatrix} 1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 1 \end{bmatrix} \begin{bmatrix} x_1(0) \\ x_2(0) \\ x_3(0) \\ \vdots \end{bmatrix} + \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} x_1(0) \\ x_2(0) \\ x_3(0) \\ \vdots \end{bmatrix} t +$$

$$\begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix}^2 \begin{bmatrix} x_1(0) \\ x_2(0) \\ x_3(0) \\ \vdots \end{bmatrix} t^2 + \cdots$$

- If ,

$$\mathbf{x}(t) = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \end{bmatrix}, \mathbf{x}(0) = \begin{bmatrix} x_1(0) \\ x_2(0) \\ x_3(0) \\ \vdots \end{bmatrix} \mathbf{A} = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix} \mathbf{I} = \begin{bmatrix} 1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 1 \end{bmatrix}$$

Solution of Homogeneous State Equations

Continuation of Solution of Vector-matrix D.E.s..

- the solution $\mathbf{x}$(t) can be written as,

$$\mathbf{x}(t) = \underbrace{\left(\mathbf{I} + \mathbf{A}t + \frac{1}{2!}(\mathbf{A})^2t^2 + \frac{1}{3!}(\mathbf{A})^3t^3 + \cdots + \frac{1}{k!}(\mathbf{A})^kt^k + \cdots\right)}\mathbf{x}(0)$$

- The expression in the under brace on the R.H.S of the last equation is an $n * n$ matrix
- It is similar to the infinite power series for a scalar exponential.It is called matrix exponential and can be written as:

$$e^{\mathbf{A}t} = \mathbf{I} + \mathbf{A}t + \tfrac{1}{2!}(\mathbf{A})^2t^2 + \tfrac{1}{3!}(\mathbf{A})^3t^3 + \cdots + \tfrac{1}{k!}(\mathbf{A})^kt^k + \cdots$$

- Thus, the solution can be written as

$$\mathbf{x}(t) = e^{\mathbf{A}t}\mathbf{x}(0)$$
$$= \boldsymbol{\phi}(t)\mathbf{x}(0)$$

where,$\boldsymbol{\phi}$(t) is called the 'State Transition Matrix'

Solution of Non Homogeneous State equations

Scalar Case

- Consider a scalar state equation,

$$\dot{x} = ax + bu$$
$$\dot{x} - ax = bu,$$

- Multiplying this equation by e^{-at} on both sides and integration between 0 and t gives,

$$x(t) = e^{at}x(0) + \int_0^t (e^{a(t-\tau)}u(\tau)d\tau)$$

Vector Case

- Consider the non homogeneous state equation described by

$$\dot{\mathbf{x}} = \mathbf{Ax} + \mathbf{Bu}$$

where, $\mathbf{x} \in \mathbb{R}^n \rightarrow$ n-vector
$\mathbf{u} \in \mathbb{R}^m \rightarrow$m-vector
$\mathbf{A} \in \mathbb{R}^{n*n} \rightarrow$ n∗n-constant matrix,
$\mathbf{B} \in \mathbb{R}^{n*m} \rightarrow$ n∗m-constant matrix,

Solution of Non Homogeneous State Equations

Continuation of Vector Case...

- The solution of $\mathbf{x}(t)$ can be written as
  $$\mathbf{x}(t) = e^{\mathbf{A}t}\mathbf{x}(0) + \int_0^t (e^{\mathbf{A}(t-\tau)}\mathbf{u}(\tau)d\tau)$$
- This equation can also be written as
  $$\mathbf{x}(t) = \boldsymbol{\phi}(t)\mathbf{x}(0) + \int_0^t (\boldsymbol{\phi}(t-\tau)\mathbf{u}(\tau)d\tau)$$
  where, $\boldsymbol{\phi}$(t)=$e^{\mathbf{A}t}$, is the State Transition Matrix'

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State Transition Matrix- 'ϕ'

Properties of State Transition Matrix- 'ϕ'

For the time invariant system, $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$

$$\phi(t) = e^{\mathbf{A}t}$$

- The properties of the State Transition Matrix are:

$$\begin{aligned}
\phi(0) &= \mathbf{I} \\
\phi^{-1}(t) &= \phi(-t) \\
\phi(t_1 + t_2) &= \phi(t_1)\phi(t_2) \\
[\phi(t)]^n &= \phi(nt) \\
\phi(t_2 - t_1)\phi(t_1 - t_0) &= \phi(t_2 - t_0)
\end{aligned}$$

State Transition Matrix- 'ϕ'

Example

Computational Methods of Matrix exponential-$e^{\mathbf{A}t}$

Methods to compute e^{At}

- Matrix exponential can be computed by
  - Numerical Methods
  - Analytic Methods

Numerical Methods

- If matrix **A** is given with all elements in numerical values, MATLAB provides a simple way to compute $e^{\mathbf{A}T}$, where T is a constant.

Analytic Methods

- Some of the Analytic methods to be discussed are given below
  - Laplace Transformation Approach
  - Diagonal Transformation
  - Cayley-Hamilton Theorem

Outline

Computational Methods of Matrix exponential-$e^{\mathbf{A}t}$

Laplace Transformation Approach

- We know that,
$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t)$$
- Applying Laplace Transformation on both sides, we get,
$$\mathbf{s}\mathbf{X}(s) - \mathbf{X}(0) = \mathbf{A}\mathbf{X}(s)$$
$$\Rightarrow (\mathbf{sI\text{-}A})\mathbf{X}(s) = \mathbf{X}(0)$$
- Pre-multiplying with $(\mathbf{sI\text{-}A})^{-1}$ on both sides and taking Inverse Laplace Transform on both sides, we get
$$\mathbf{x}(t) = \mathcal{L}^{-1}((\mathbf{sI\text{-}A})^{-1})\mathbf{x}(0)$$
- We know that,
$$\mathbf{x}(t) = e^{\mathbf{A}t}\mathbf{x}(0)$$

Comparing equations, (1) and (2) we can write
$$e^{\mathbf{A}t} = \mathcal{L}^{-1}((\mathbf{sI\text{-}A})^{-1}))$$

Computational Methods of Matrix exponential-$e^{\mathbf{A}t}$

Example of Laplace Transformation approach

- Consider the following matrix $\mathbf{A} = \begin{bmatrix} -1 & 0 \\ -3 & -2 \end{bmatrix}$

$$\begin{aligned}
\mathbf{sI\text{-}A} &= \begin{bmatrix} \mathbf{s} & 0 \\ 0 & \mathbf{s} \end{bmatrix} - \begin{bmatrix} -1 & -0 \\ -3 & -2 \end{bmatrix} \\
\Rightarrow (\mathbf{sI\text{-}A})^{-1} &= \begin{bmatrix} s+1 & 0 \\ 3 & s+2 \end{bmatrix}^{-1} \\
\Rightarrow (\mathbf{sI\text{-}A})^{-1} &= \begin{bmatrix} \frac{1}{s+1} & 0 \\ \frac{-3}{(s+1)(s+2)} & \frac{1}{s+2} \end{bmatrix}
\end{aligned}$$

Applying Inverse Laplace Transformation on both sides we get,

$$e^{\mathbf{A}t} = \mathcal{L}^{-1}((\mathbf{sI\text{-}A})^{-1}) = \begin{bmatrix} e^{-t} & 0 \\ -3e^{-t} + 3e^{-2t} & e^{-2t} \end{bmatrix}$$

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COMPUTATIONAL METHODS OF MATRIX EXPONENTIAL-$e^{\mathbf{A}t}$

DIAGONAL TRANSFORMATION

- Matrix **A** is diagonalized using a diagonalizing matrix **P**
- The resultant matrix is given by

$$e^{\mathbf{A}t} = \mathbf{P}e^{\mathbf{\Lambda}t}\mathbf{P}^{-1} = \mathbf{P}\begin{bmatrix} e^{\lambda_1 t} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & e^{\lambda_n t} \end{bmatrix}\mathbf{P}^{-1}$$

- If matrix **A** can be transformed into Jordan Canonical form, then $e^{\mathbf{A}t}$ can be given by

$$e^{\mathbf{A}t} = \mathbf{S}e^{\mathbf{J}t}\mathbf{S}^{-1}$$

  where, **S** is a transformation matrix that transforms matrix **A** into Jordan canonical form **J**

Computational Methods of Matrix exponential-$e^{\mathbf{A}t}$

Example of Diagonal Transformation

- Let, matrix $\mathbf{A} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}$
- characteristic equation of this matrix is given by $\lambda^2 - 3\lambda + 2$
  Eigenvalues of matrix A are $\lambda_1 = -1, \lambda_2 = -2$
  The corresponding eigenvectors of the eigenvalues λ_1 and λ_2 are $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ respectively
- The modal matrix formed by these eigenvectors is given by $\mathbf{P} = \begin{bmatrix} 1 & 1 \\ -1 & 2 \end{bmatrix}$
- the diagonal matrix $\mathbf{\Lambda}$ is obtained by the transformation $\mathbf{P}^{-1}\mathbf{AP}$

$$\mathbf{\Lambda} = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix}$$

$$\therefore e^{\mathbf{J}t} = e^{\mathbf{\Lambda}t} = \begin{bmatrix} e^{-t} & 0 \\ 0 & e^{-2t} \end{bmatrix}$$

Computational Methods of Matrix exponential-$e^{\mathbf{A}t}$

Continuation of Example of Diagonal Transformation

- Having transformed Matrix **A** into Jordan Canonical form, Matrix exponential $e^{\mathbf{A}t}$ can be obtained by the transformation $\mathbf{P}e^{\mathbf{J}t}\mathbf{P}^{-1}$

$$
\begin{aligned}
e^{\mathbf{A}t} &= \begin{bmatrix} 1 & 1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} e^{-t} & 0 \\ 0 & e^{-2t} \end{bmatrix} \begin{bmatrix} 2 & 1 \\ -1 & -1 \end{bmatrix} \\
&= \begin{bmatrix} 2e^{-t} - e^{-2t} & e^{-t} - e^{-2t} \\ -2e^{-t} 2e^{-2t} & -e^{-t} + 2e^{-2t} \end{bmatrix}
\end{aligned}
$$

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Computational Methods of Matrix exponential-$e^{\mathbf{A}t}$

Cayley-Hamilton Theorem Approach

- For large systems, this method is far more convenient computationally as compared to the other two methods discussed earlier.

Cayley-Hamilton theorem

- Statement:Every square matrix **A** satisfies its own characteristic equation.

  If,$q(\lambda) = \lambda^n + a_1\lambda^{n-1} + \cdots + a_{n-1}\lambda + a_n = 0$ is the characteristic equation of **A**, then

$$q(\mathbf{A}) = \mathbf{A}^n + a_1\mathbf{A}^{n-1} + \cdots + a_{n-1}\mathbf{A} + a_n = 0$$

Computational Methods of Matrix exponential-$e^{\mathbf{A}t}$

Cayley-Hamilton theorem

- Let $\lambda_1, \lambda_2, \cdots, \lambda_n$ be the eigenvalues of the matrix $\mathbf{A}$
- Consider the scalar polynomial $f(\lambda) = k_0 + k_1\lambda + k_2\lambda^2 + \cdots + k_n\lambda^n + \cdots$, where λ is the eigenvalue of the matrix.
- The matrix polynomial $f(\mathbf{A}) = k_0\mathbf{I} + k_1\mathbf{A} + k_2\mathbf{A}^2 + \cdots + k_n\mathbf{A}^n + \cdots$ an be computed by considering the scalar polynomial $f(\lambda)$
- Dividing $f(\lambda)$ by $q(\lambda)$ we get $\frac{f(\lambda)}{q(\lambda)} = Q(\lambda) + \frac{R(\lambda)}{q(\lambda)}$
  $\therefore f(\lambda) = Q(\lambda)q\lambda + R(\lambda)$
  where, $R(\lambda) = \alpha_0 + \alpha_1\lambda + \alpha_2\lambda^2 + \cdots + \alpha_{n-1}\lambda^{n-1}$ is the remainder polynomial
- For $\lambda = \lambda_1, \lambda_2 \cdots, \lambda_n$ (for eigenvalues) $q(\lambda) = 0$

$$\therefore f(\lambda_i) = R(\lambda_i); i = 1, 2, 3, \cdots$$

Computational Methods of Matrix exponential-$e^{\mathbf{A}t}$

Cayley-Hamilton theorem

- The coefficients of the remainder polynomial $\alpha_0, \alpha_1, \cdots, \alpha_{n-1}$ can be obtained by substituting $\lambda = \lambda_1, \lambda_2, \cdots, \lambda_n$ in the relation $f(\lambda_i) = R(\lambda_i)$
- Replacing λ with matrix $\mathbf{A}$ we get,

$$\begin{aligned} f(\mathbf{A}) &= R(\mathbf{A}) \\ &= \alpha_0\mathbf{I} + \alpha_1\mathbf{A} + \cdots + \alpha_{n-1}\boldsymbol{A}^{\boldsymbol{n-1}} \end{aligned}$$

Computational Methods of Matrix exponential-$e^{\mathbf{A}t}$

Cayley-Hamilton theorem

Procedure to compute $e^{\mathbf{A}t}$:

Step-1:

- Find the eigenvalues of matrix $\mathbf{A}$

Step-2:

- Case-1: If all the eigenvalues are distinct, the coefficients $\alpha_0, \alpha_1, \cdots, \alpha_{n-1}$ can be obtained by solving $'n'$ simultaneous equations given by $f(\mathbf{A}) = R(\mathbf{A})$
- Case-2: If $\mathbf{A}$ possess an eigenvalue λ_K of order $'m'$ then,
  - Only one independent equation can be obtained by substituting λ_k in the equation $f(\mathbf{A}) = R(\mathbf{A})$
  - The remaining $m-1$ linear equations can be obtained by differentiating $f(\lambda) = R(\lambda)$ on both sides

$$\therefore \left.\frac{d^j f(\lambda)}{d\lambda^j}\right|_{\lambda=\lambda_k} = \left.\frac{d^j R(\lambda)}{d\lambda^j}\right|_{\lambda=\lambda_k}; j = 0, 1, 2 \cdots, m-1$$

Step-3:

- The required result is obtained by substituting the values of $\alpha_0, \alpha_1, \cdots, \alpha_{n-1}$ in $f(\mathbf{A}) = \alpha_0\mathbf{I} + \alpha_1\mathbf{A} + \cdots + \alpha_{n-1}\boldsymbol{A}^{\boldsymbol{n-1}}$

Computational Methods of Matrix exponential-$e^{\mathbf{A}t}$

Example of Cayley-Hamilton theorem Approach

Consider the matrix, $\mathbf{A}=\begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix}$

Step-1:

- Eigenvalues of the matrix are $\lambda_1 = \lambda_2 = -1$

Step-2:

- Since the eigenvalues equal and of order 2
- $\therefore 2 - 1 = 1$ equation can be obtained by differentiating the coefficients of $f(\mathbf{A}) = R(\mathbf{A})$
- Another equation is obtained by substituting $\lambda = -1$ directly in the equation $f(\mathbf{A}) = R(\mathbf{A})$

Computational Methods of Matrix exponential-$e^{\mathbf{A}t}$

Example of Cayley-Hamilton theorem Approach

- Since $\mathbf{A}$ is of second-order, The polynomial $R(\lambda)$ will be of the form $\alpha_0 + \alpha_1\lambda$
- The coefficients of α_0, α_1 can be obtained as follows:

$$\begin{aligned} f(\lambda) &= e^{\lambda t} = \alpha_0 + \alpha_1\lambda \\ \therefore f(-1) &= e^{-t} = \alpha_0 - \alpha_1 \qquad (4) \\ \frac{d}{d\lambda}f(\lambda)_{\lambda=-1} &= \frac{d}{d\lambda}e^{\lambda t}_{\lambda=-1} = \frac{d}{d\lambda}(\alpha_0 + \alpha_1\lambda) \\ \Rightarrow te^{-t} &= \alpha_1 \end{aligned}$$

- Substituting the value of α_1 in equation(4) we get $\alpha_0 = (1+t)e^{-t}$

Step-3:

- The required result is $f(\mathbf{A}) = e^{\mathbf{A}t} = \alpha_0\mathbf{I} + \alpha_1\mathbf{A}$

$$= \begin{bmatrix} (1+t)e^{-t} & te^{-t} \\ -te^{-t} & (1-t)e^{-t} \end{bmatrix}$$