Control systems

Lecture-4 Stability

V. Sankaranarayanan

Outline

1. Theory

Outline

Concept of Stability

Zero State Response:
The zero-state response is due to the input only; all the initial conditions of the system are zero. It is also known as forced response.

Concept of Stability

Zero State Response:
The zero-state response is due to the input only; all the initial conditions of the system are zero. It is also known as forced response.

Zero Input Response:
The zero-input response is due to the initial conditions only,all the inputs are zero. It is also known as natural response.

Total Response=Forced Response+Natural Response

Concept of Stability

Zero State Response:
The zero-state response is due to the input only; all the initial conditions of the system are zero. It is also known as forced response.

Zero Input Response:
The zero-input response is due to the initial conditions only,all the inputs are zero. It is also known as natural response.

Total Response=Forced Response+Natural Response

Stability Definition

A linear time invariant system is stable if the natural response approaches to zero as the time approaches to infinity.

BIBO Stability

Definition

A system is said to be Bounded Input Bounded Output(BIBO) stable if every bounded input yield bounded output.

BIBO Stability

Let $u(t)$,$y(t)$,$g(t)$ be the input, output,impulse response of a system. As we know that

$$y(t) = \int_0^\infty u(t-\tau)g(\tau)d\tau$$

$$|y(t)| = \left|\int_0^\infty u(t-\tau)g(\tau)d\tau\right| \qquad \leq \int_0^\infty |u(t-\tau)||g(\tau)|d\tau$$

Let u(t) is bounded

$$|u(t)| \leq M \qquad \text{M is a positive finite number}$$

$$|y(t)| \leq M\int_0^\infty g(\tau)d\tau$$

$y(t)$ to be bounded

$$|y(t)| \leq K < \infty$$

$$M\int_0^\infty g(\tau)d\tau \leq K < \infty$$

$$\int_0^\infty g(\tau)d\tau \leq N < \infty$$

BIBO Stability

Condition

A system is said to be stable if the area under the $|g(t)|$ is finite.

Stability: Pole Position

Stable System

A stable system have close loop transfer function with poles only in the left half of s-plane.

Stability: Pole Position

Stable System

A stable system have close loop transfer function with poles only in the left half of s-plane.

Unstable System

Unstable system has closed loop transfer function with atleast one pole on the right half of s-plane and/or pole of multiplicity greater than 1 on the imaginary axis giving rise to response of form $t^n \cos(\omega t + \phi)$

Stability: Pole Position

Stable System

A stable system have close loop transfer function with poles only in the left half of s-plane.

Unstable System

Unstable system has closed loop transfer function with atleast one pole on the right half of s-plane and/or pole of multiplicity greater than 1 on the imaginary axis giving rise to response of form $t^n \cos(\omega t + \phi)$

Marginally Stable System

A marginally system has closed loop transfer function with poles only on the imaginary axis with multiplicity 1.

Stability: Pole Position

Stable System

A stable system have close loop transfer function with poles only in the left half of s-plane.

Unstable System

Unstable system has closed loop transfer function with atleast one pole on the right half of s-plane and/or pole of multiplicity greater than 1 on the imaginary axis giving rise to response of form $t^n \cos(\omega t + \phi)$

Marginally Stable System

A marginally system has closed loop transfer function with poles only on the imaginary axis with multiplicity 1.

Routh's Stability Criterion

Equation

Consider a characteristic equation system of the form

$$a_n s^n + a_{n-1} s^{n-1} + ... + a_1 s + a_0 = 0$$

Routh's Stability Criterion

Equation

Consider a characteristic equation system of the form

$$a_n s^n + a_{n-1} s^{n-1} + ... + a_1 s + a_0 = 0$$

All the coefficient must be of same sign and nonzero. This is a **necessary condition** for the roots of equation to have negative real part.

Routh's Stability Criterion

Equation

Consider a characteristic equation system of the form

$$a_n s^n + a_{n-1} s^{n-1} + ... + a_1 s + a_0 = 0$$

Generation of Routh Table: Consider $a_4 s^4 + a_3 s^3 + a_2 s^2 + a_1 s + a_0 = 0$

s^4	a_4	a_2	a_0
s^3	a_3	a_1	0
s^2	$\frac{a_3 a_2 - a_1 a_4}{a_3} = b_1$	$\frac{a_3 a_0 - 0 \times a_4}{a_3} = b_2$	0
s^1	$\frac{b_1 a_1 - b_2 a_3}{b_1} = c_1$	$\frac{0 \times b_2 - 0 \times a_1}{b_1} = 0$	0
s^0	$\frac{c_1 b_2 - 0 \times b_1}{c_1}$	0	0

Routh's Stability Criterion

Stability Criterion

The number of roots of equation with positive real parts is equal to the number of **change in sign** of coefficients of first column of table.

Generation of Routh Table: Consider $a_4s^4 + a_3s^3 + a_2s^2 + a_1s + a_0 = 0$

s^4	a_4	a_2	a_0
s^3	a_3	a_1	0
s^2	$\dfrac{a_3a_2 - a_1a_4}{a_3} = b_1$	$\dfrac{a_3a_0 - 0 \times a_4}{a_3} = b_2$	0
s^1	$\dfrac{b_1a_1 - b_2a_3}{b_1} = c_1$	$\dfrac{0 \times b_2 - 0 \times a_1}{b_1} = 0$	0
s^0	$\dfrac{c_1b_2 - 0 \times b_1}{c_1}$	0	0

Example

Consider a characteristic equation:

$$s^3 + 10s^2 + 31s + 1030 = 0$$

Routh Table:

s^3	1	31	0
s^2	~~10~~ 1	~~1030~~ 103	0
s^1	-72	0	0
s^0	103	0	0

- There is two sign change. So there are two pole in right half of s-plane.
- Any row can be multiplied with a constant value. Only signs are important not absolute values

Example

Consider a characteristic equation:

$$s^3 + 10s^2 + 31s + 1030 = 0$$

Routh Table:

s^3	1	31	0
s^2	~~10~~ 1	~~1030~~ 103	0
s^1	-72	0	0
s^0	103	0	0

- There is two sign change. So there are two pole in right half of s-plane.
- Any row can be multiplied with a constant value. Only signs are important not absolute values
- Roots: $-13.4136, 1.7068 \pm 8.5950i$

Special Cases

Zero in First Column

A zero in the first column is replaced by ϵ. The value of ϵ is allowed to appraoch to zero from positive side and signs are evaluated.

Special Cases

Consider

$$s^5 + 2s^4 + 3s^3 + 6s^2 + 5s + 3 = 0$$

Routh Table:

s^5	1	$+$	3	5
s^4	2	$+$	6	3
s^3	$\not{0}\ \ \epsilon$	$+$	3.5	0
s^2	$\frac{6\epsilon - 7}{\epsilon}$	$-$	3	0
s^1	$\frac{42\epsilon - 49 - 6\epsilon^2}{12\epsilon - 14}$	$+$	0	0
s^0	3	$+$	0	0

- Two sign change indicates two roots with positive real part.
- Roots: $0.3429 \pm 1.5083i, -1.6681, -0.5088 \pm 0.7020i$

Entire Row Zero

Zeros in a Row

If there is an even polynomial that is a factor of the original polynomial an entire row consist of zeros in will appear in Routh Table

Entire Row Zero

Consider:
$s^5 + 7s^4 + 6s^3 + 42s^2 + 8s + 56 = 0$

s^5	1	6	8
s^4	$\not{7}$ 1	$\not{42}$ 6	$\not{56}$ 8
s^3	$\not{0}$ $\not{4}$ 1	$\not{0}$ $\not{12}$ 3	$\not{0}$ 0
s^2	3	8	0
s^1	$\frac{1}{3}$	0	0
s^0	8	0	0

Roots: $-7, \pm 2i, \pm 1.4142i$

Entire Row Zero

Formation of Auxiliary Equation: Take coefficients of row above zero row(3rd) to form a polynomial starting with power of s as per label and continue by skipping every other power of s.

$$P(s) = s^4 + 6s^2 + 8$$

Differentiate it with s and replace the zero row with obtained coefficients

$$\frac{dP}{ds} = 4s^3 + 12s$$

Consider:

$$s^5 + 7s^4 + 6s^3 + 42s^2 + 8s + 56 = 0$$

s^5	1	6	8
s^4	~~7~~ 1	~~42~~ 6	~~56~~ 8
s^3	~~0~~ ~~4~~ 1	~~0~~ ~~12~~ 3	~~0~~ 0
s^2	3	8	0
s^1	$\frac{1}{3}$	0	0
s^0	8	0	0

Roots: $-7, \pm 2i, \pm 1.4142i$

Entire Row Zero

$$s^5 + 7s^4 + 6s^3 + 42s^2 + 8s + 56$$
$$= (s+7)(s^4 + 6s^2 + 8) = (s+7)P(s)$$
$$= Q(s)P(s)$$

- Row above zero row gives information on Q(s)
- Row below zero row gives information on P(s)
- The roots of $P(s)$ is symmetric with respect to origin.

Consider:
$s^5+7s^4+6s^3+42s^2+8s+56=0$

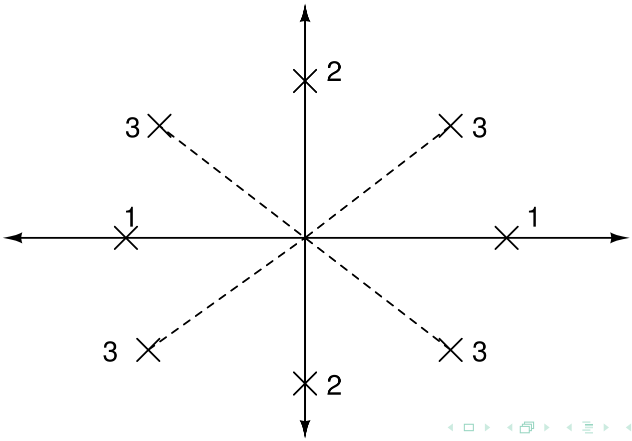
s^5	1	6	8
s^4	~~7~~ 1	~~42~~ 6	~~56~~ 8
s^3	~~0~~ ~~4~~ 1	~~0~~ ~~12~~ 3	~~0~~ 0
s^2	3	8	0
s^1	$\frac{1}{3}$	0	0
s^0	8	0	0

Roots: $-7, \pm 2i, \pm 1.4142i$

More on Entire Row Zero

Poles Symmetric with Origin

1. The roots are symmetrical and real
2. The roots are symmetrical and imaginary
3. The roots are quadrantal



Hurwitz stability criterion

Consider a characteristic equation of a system in the form

$$a_0 s^n + a_1 s^{n-1} + \dots\dots\dots + a_{n-11} s + a_n = 0$$

$$H = \begin{bmatrix} a_1 & a_3 & a_5 & \dots & \dots & \dots & 0 & 0 & 0 \\ a_0 & a_2 & a_4 & & & & \vdots & \vdots & \vdots \\ 0 & a_1 & a_3 & \ddots & & & \vdots & \vdots & \vdots \\ \vdots & a_0 & a_2 & & \ddots & & 0 & \vdots & \vdots \\ \vdots & 0 & a_1 & & & \ddots & a_n & \vdots & \vdots \\ \vdots & \vdots & a_0 & & & & a_{n-1} & 0 & \vdots \\ \vdots & \vdots & 0 & & & & a_{n-2} & a_n & \vdots \\ \vdots & \vdots & \vdots & & & & a_{n-3} & a_{n-1} & 0 \\ \vdots & \vdots & \vdots & \dots & \dots & \dots & a_{n-4} & a_{n-2} & a_n \end{bmatrix}$$

Hurwitz stability criterion

Consider a characteristic equation of a system in the form

$$a_0 s^n + a_1 s^{n-1} + \dots\dots\dots + a_{n-11} s + a_n = 0$$

$$H = \begin{bmatrix} a_1 & a_3 & a_5 & \dots & \dots & \dots & 0 & 0 & 0 \\ a_0 & a_2 & a_4 & & & & \vdots & \vdots & \vdots \\ 0 & a_1 & a_3 & \ddots & & & \vdots & \vdots & \vdots \\ \vdots & a_0 & a_2 & & \ddots & & 0 & \vdots & \vdots \\ \vdots & 0 & a_1 & & & \ddots & a_n & \vdots & \vdots \\ \vdots & \vdots & a_0 & & & & a_{n-1} & 0 & \vdots \\ \vdots & \vdots & 0 & & & & a_{n-2} & a_n & \vdots \\ \vdots & \vdots & \vdots & & & & a_{n-3} & a_{n-1} & 0 \\ \vdots & \vdots & \vdots & \dots & \dots & \dots & a_{n-4} & a_{n-2} & a_n \end{bmatrix}$$

The roots of the equations have negative real parts if and only if all the principal minor of H is positive.

$$|a_1| > 0 \qquad \begin{vmatrix} a_1 & a_3 \\ a_0 & a_2 \end{vmatrix} > 0 \qquad \begin{vmatrix} a_1 & a_3 & a_5 \\ a_0 & a_2 & a_4 \\ 0 & a_1 & a_3 \end{vmatrix} > 0$$

Example

Consider a characteristic equation:

$$s^3 + 10s^2 + 31s + 1030 = 0$$

$$H = \begin{vmatrix} 10 & 1030 & 0 \\ 1 & 10 & 0 \\ 0 & 10 & 1030 \end{vmatrix}$$

Example

Consider a characteristic equation:

$$s^3 + 10s^2 + 31s + 1030 = 0$$

$$H = \begin{vmatrix} 10 & 1030 & 0 \\ 1 & 10 & 0 \\ 0 & 10 & 1030 \end{vmatrix}$$

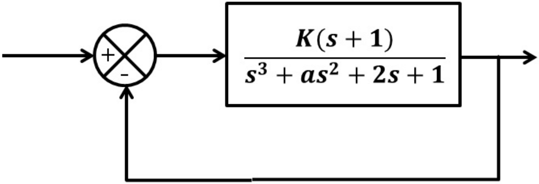
Now

$$|10| > 0 \qquad \begin{vmatrix} 10 & 1030 \\ 1 & 10 \end{vmatrix} = -930 < 0 \qquad \begin{vmatrix} 10 & 1030 & 0 \\ 1 & 10 & 0 \\ 0 & 10 & 1030 \end{vmatrix} = -957900$$

So the system is not stable.

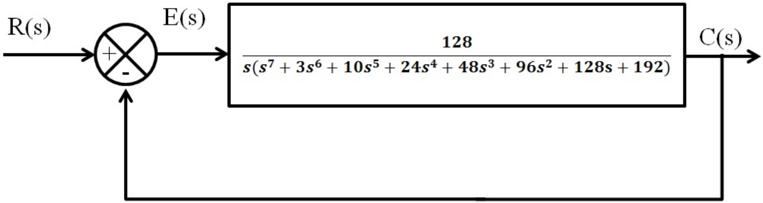
Assignment

1. Find the values of K and a for the following system to oscillate at a frequency of 2 rad/ sec.



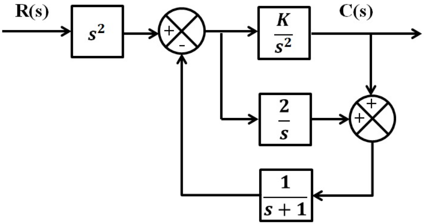
Assignment

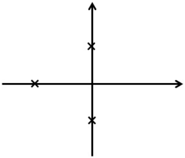
2. Find the number of poles in the left half-plane, the right half-plane, and on the $j\omega$-axis for the system. Draw conclusions about the stability of the closed-loop system.



Assignment

3. Find the value of K in the system of figure. that will place the closed loop poles as shown in figure.





Assignment

4. For the unity feedback system with

$$G(s) = \frac{K(S+2)}{(s^2+1)(s+4)(s-1)}$$

Find the range of K for which there will be only two closed-loop, right-half-plane poles.

Assignment

5. For the unity feedback system with

$$G(s) = \frac{K}{(s+1)^3(s+4)}$$

a. Find the range of K for the stability.
b. Find the frequency of oscillation when the system is marginally stable.

Assignment

6. Utilizing the Routh-Hurwitz criterion, determine the stability of the polynomial $s^5 + s^4 + 2s^3 + s + 5$. Determine the number of roots, if any, in the right-hand plane. Also when it is adjustable, determine the range of K that results in a stable system.

Assignment

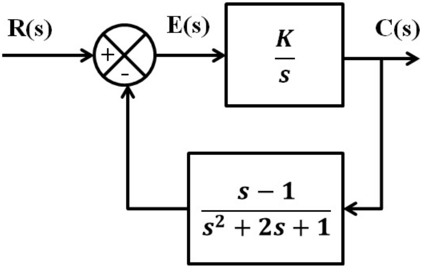
7. Utilizing the Routh-Hurwitz criterion, determine the stability of the polynomial $s^5 + s^4 + 2s^3 + s^2 + s + K$. Determine the number of roots, if any, in the right-hand plane. Also when it is adjustable, determine the range of K that results in a stable system.

Assignment

8. A feedback control system has a characteristic equation $s^3 + (1 + k)s^2 + 10s + (5 + 15K){=}0$. The parameter K must be positive. What is the maximum value K can assume before the system becomes unstable? When K is equal to the maximum value, the system oscillates. Determine the frequency of oscillation.

Assignment

9. Find the range of K to keep the system shown in the figure stable.



Assignment

10. For the block diagram shown in the given figure , the limiting value of K for the stability of inner loop is found to be $X < K < Y$. Find the condition for overall system to be stable.