Modern Control systems

Lecture-5 State Space to Transfer Function

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Outline

State Space to Transfer Function

Deriving Transfer Function from State Space

- Previously we have seen different methods of obtaining State-Space from Transfer function.
- The process of converting Transfer Function to State-Space form is **NOT** unique .
- Deriving Transfer function model from a State-Space model is **UNIQUE.**
- We know that,

$$\begin{aligned} \dot{x} &= \boldsymbol{A}x + \boldsymbol{B}u \\ y &= \boldsymbol{C}x + \boldsymbol{D}u \end{aligned}$$

- Applying Laplace Transform with zero initial conditions we get,

$$\begin{aligned} \boldsymbol{sX}(s) &= \boldsymbol{AX}(s) + \boldsymbol{BU}(s) \\ \boldsymbol{Y}(s) &= \boldsymbol{CX}(s) + \boldsymbol{D}(s) \end{aligned}$$

State Space to Transfer Function

Deriving Transfer Function from State Space

- The state equation can also be written as

$$(\boldsymbol{sI} - \boldsymbol{A})\boldsymbol{X}(s) = \boldsymbol{BU}(s)$$

- Pre multiplying on both sides by $(\boldsymbol{sI} - \boldsymbol{A})^{-1}$ we get,

$$\boldsymbol{X}(s) = (\boldsymbol{sI} - \boldsymbol{A})^{-1}\boldsymbol{BU}(s)$$

- Substituting $\boldsymbol{X}(s)$ in the output equation we get,

$$\begin{aligned} \boldsymbol{Y}(s) &= \boldsymbol{C}(\boldsymbol{sI} - \boldsymbol{A})^{-1}\boldsymbol{BU}(s) + \boldsymbol{DU}(s) \\ \Rightarrow \boldsymbol{Y}(s) &= \underbrace{\left[\boldsymbol{C}(\boldsymbol{sI} - \boldsymbol{A})^{-1}\boldsymbol{B} + \boldsymbol{D}\right]}_{\text{Transfer Function Matrix } \boldsymbol{T}(s)} \boldsymbol{U}(s) \end{aligned}$$

State Space to Transfer Function

Example on State Space to Transfer Function

- Consider the State Space representation,

$$\begin{aligned}\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix} &= \begin{bmatrix}0 & 1\\ -15 & -2\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} + \begin{bmatrix}0\\ 1\end{bmatrix}\\ y &= \begin{bmatrix}-66 & -3\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} + [5]]u\end{aligned}$$

- Here, $A = \begin{bmatrix}0 & 1\\ -15 & -2\end{bmatrix}, B = \begin{bmatrix}0\\ 1\end{bmatrix}, C = \begin{bmatrix}-66 & -3\end{bmatrix}, D = [5]$
- We know that, Transfer Function Matrix

$$\boldsymbol{T}(s) = \frac{\boldsymbol{Y}(s)}{\boldsymbol{U}(s)} = \left[\boldsymbol{C}(s\boldsymbol{I} - \boldsymbol{A})^{-1}\boldsymbol{B}\boldsymbol{U} + \boldsymbol{D}\right]$$

State Space to Transfer Function

Example on State Space to Transfer Function

$$
\begin{aligned}
(s\boldsymbol{I} - \boldsymbol{A}) &= \begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ -15 & -2 \end{bmatrix} \\
&= \begin{bmatrix} s & -1 \\ 15 & s+2 \end{bmatrix} \\
(s\boldsymbol{I} - \boldsymbol{A})^{-1} &= \frac{1}{s^2 + 2s + 17} \begin{bmatrix} s+2 & 1 \\ -15 & s \end{bmatrix}
\end{aligned}
$$

$$
\begin{aligned}
\left[\boldsymbol{C}(s\boldsymbol{I} - \boldsymbol{A})^{-1}\boldsymbol{B}\right] + \boldsymbol{D} &= \frac{1}{s^2 + 2s + 17} \left(\begin{bmatrix} -66 & -3 \end{bmatrix} \begin{bmatrix} s+2 & 1 \\ -15 & s \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right) + [5] \\
&= \frac{-66 - 3s}{s^2 + 2s + 17} + 5 \\
&= \frac{5s^2 + 7s + 9}{s^2 + 2s + 15}
\end{aligned}
$$

State Space to Transfer Function

Example on State Space to Transfer Function

$$
\begin{aligned}
&= \frac{-66-3s}{s^2+2s+17}+5 \\
&= \frac{5s^2+7s+9}{s^2+2s+15}
\end{aligned}
$$

- Students are advised to compare this result with that of Example-2 of Canonical Form-I in Lecture -4

Similarity Transformation

- The state space equations of a system is given as

$$\begin{aligned} \dot{x} &= \boldsymbol{A}x + \boldsymbol{B}u \\ y &= \boldsymbol{C}x + \boldsymbol{D}u \end{aligned}$$

- Choose any non-singular matrix '$\boldsymbol{T}$' such that
  $z = \boldsymbol{T}x \Rightarrow x = \boldsymbol{T}^{-1}z$ and $\dot{x} = \boldsymbol{T}^{-1}\dot{z}$
- Substituting in the system equations

$$\begin{aligned} \dot{z} &= \boldsymbol{TAT^{-1}}z + \boldsymbol{TB}u \\ y &= \boldsymbol{CT^{-1}}z + \boldsymbol{D}u \end{aligned}$$

- It can be written as

$$\begin{aligned} \dot{z} &= \boldsymbol{A_z} + \boldsymbol{D_z}u \\ y &= \boldsymbol{C_z}z + \boldsymbol{D_z}u \end{aligned}$$

where,
$\boldsymbol{A_z} = \boldsymbol{TAT^{-1}}$; $\boldsymbol{B_z} = \boldsymbol{TB}$; $\boldsymbol{C_z} = \boldsymbol{CT^{-1}}$; $\boldsymbol{D_z} = \boldsymbol{D}$

Similarity Transformation

Example on Similarity transform

- Consider the matrix $\boldsymbol{A} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}$ with eigenvalues $-2, -1$
- Let, $\boldsymbol{A_z}$ be the transformed matrix
- We know that, $\boldsymbol{A_z} = \boldsymbol{T^{-1}AT}$, where $\boldsymbol{T}$ is a non-singular, n $\times$ n matrix.
- The key to similarity transformation is finding the matrix $\boldsymbol{T}$ corresponding to the desired canonical transformation.
- Let, $\boldsymbol{T} = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}$ and the desired transformation be the Diagonal Transformation.
- Therefore, the transformed matrix $\boldsymbol{A_z}$ becomes

$$\boldsymbol{A_z} = \begin{bmatrix} -2 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}^{-1} \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} \begin{bmatrix} -2 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}^{-1}$$

Similarity Transformation

Example on Similarity transform

$$\Rightarrow \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} = \frac{1}{|\boldsymbol{T}|} \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} \begin{bmatrix} -2 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} b_{22} & -b_{12} \\ -b_{21} & b_{11} \end{bmatrix}$$

$$\left(\boldsymbol{T^{-1}} = \frac{1}{|\boldsymbol{T}|} \begin{bmatrix} b_{22} & -b_{12} \\ -b_{21} & b_{11} \end{bmatrix}, \text{ where } |\boldsymbol{T}| = b_{11}b_{22} - b_{12}b_{21} \right)$$

$$\begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} = \frac{1}{b_{11}b_{22} - b_{12}b_{21}} \begin{bmatrix} -2b_{11}b_{22} + b_{12}b_{21} & 3b_{11}b_{12} \\ -b_{21}b_{22} & 2b_{12}b_{21} - b_{11}b_{22} \end{bmatrix}$$

- We know that, if two matrices are equal then their corresponding elements are equal.
- Therefore, equating corresponding elements, we get,

$$\begin{aligned} -2b_{11}b_{22} + b_{12}b_{21} &= 0 \\ 3b_{11}b_{12} &= b_{11}b_{22} - b_{12}b_{21} \\ -b_{21}b_{22} &= -2(b_{11}b_{22} - b_{12}b_{21}) \\ 2b_{12}b_{21} - b_{11}b_{22} &= -3(b_{11}b_{22} - b_{12}b_{21}) \end{aligned}$$

Similarity Transformation

Example on Similarity transform

- On solving these equations we arrive upon

$$\begin{aligned} b_{21} &= -2b_{11} \\ b_{22} &= -b_{12} \end{aligned}$$

- If $b_{11} = b_{12} = 1$, we get $b_{21} = -2$, $b_{22} = -1$
- Therefore, $\boldsymbol{T} = \begin{bmatrix} 1 & 1 \\ -2 & -1 \end{bmatrix}$ and $\boldsymbol{T^{-1}} = \begin{bmatrix} -1 & -1 \\ 2 & 1 \end{bmatrix}$

Similarity Transformation

Example on Similarity transform

Verification:

- We know that,

$$\begin{aligned} \boldsymbol{A_z} &= \boldsymbol{T}^{-1} \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} \boldsymbol{T} \\ \boldsymbol{A_z} &= \begin{bmatrix} -1 & -1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -2 & -1 \end{bmatrix} \\ &= \begin{bmatrix} -2 & 0 \\ 0 & -1 \end{bmatrix} \end{aligned}$$

- Hence, the obtained ,matrix $\boldsymbol{T}$ is the desired matrix for transforming matrix $\boldsymbol{A}$ into Diagonal Canonical form.
- Students are advised to compare $\boldsymbol{T}$ with the modal matrix $\boldsymbol{P}$ formed by the eigenvectors of $\boldsymbol{A}$ which is used to transform matrix $\boldsymbol{A}$ into a Diagonal matrix.