Control systems

Frequency domain analysis: Polar Plot

V. Sankaranarayanan
It is a plot of the magnitude of $G(j\omega)$ versus the phase angle of $G(j\omega)$ on polar coordinates as $\omega$ is varied from zero to infinity.

In polar plots a positive (negative) phase angle is measured counterclockwise (clockwise) from the positive real axis.
**Integral and derivative factors**

- \( G(j\omega) = \frac{1}{j\omega} = -j \frac{1}{\omega} = \frac{1}{\omega} \angle -90^\circ \)
- The Polar plot of \( \frac{1}{j\omega} \) is the negative imaginary axis
The Polar plot of $j\omega$ is the positive imaginary axis
First order factors

- \( G(j\omega) = \frac{1}{1+j\omega T} = \frac{1}{\sqrt{1+\omega^2 T^2}} \angle -\tan^{-1} \omega T \)
- \( G(j0) = 1 \angle 0^\circ, \ G(j \frac{1}{T}) = \frac{1}{\sqrt{2}} \angle 45^\circ \)
- \( \omega \to \infty, \ |G(j\omega)| \to 0 \) and \( \angle G(j\omega) \to -90^\circ \)

\[
G(s) = X + jY \\
\frac{1}{1 + j\omega T} = \frac{1 - j\omega T}{1 + \omega^2 T^2} \\
G(s) = \frac{1}{1 + \omega^2 T^2} - j \frac{\omega T}{1 + \omega^2 T^2}
\]

\[
X = \frac{1}{1 + \omega^2 T^2} \quad Y = -\frac{\omega T}{1 + \omega^2 T^2}
\]

\[
\left(X - \frac{1}{2}\right)^2 + Y^2 = \left(\frac{1}{2}\right)^2
\]

So the polar plot is a semi circle.
\[ G(s) = \frac{1}{1 + j\omega T} \]
First order factors

- \[ G(j\omega) = 1 + j\omega T \]
- It is simply the upper half of the straight line passing through point \((1, 0)\) in the complex plane and parallel to the imaginary axis.
Quadratic factors

\[ G(j\omega) = \frac{1}{1 + 2\zeta(j\frac{\omega}{\omega_n}) + (j\frac{\omega}{\omega_n})^2} \]

For \( \zeta > 0 \)

\[ \lim_{\omega \to 0} G(j\omega) = 1 \angle 0 \]

\[ \lim_{\omega \to \infty} G(j\omega) = 0 \angle -180 \]

The polar plot of this sinusoidal transfer function starts at 1\( \angle 0 \) and ends at 0\( \angle -180 \) as \( \omega \) increases from zero to infinity.

The high-frequency portion of \( G(j\omega) \) is tangent to the negative real axis.

For the underdamped case \( \omega = \omega_n \), the phase angle is \(-90\)°.

In the polar plot, the frequency point whose distance from the origin is maximum corresponds to the resonant frequency \( \omega_r \).
Quadratic factors

- $G(j\omega) = 1 + 2\zeta(j\frac{\omega}{\omega_n}) + (j\frac{\omega}{\omega_n})^2$
- $= \left(1 - \frac{\omega^2}{\omega_n^2}\right) + j\left(\frac{2\zeta\omega}{\omega_n}\right)$
- $\lim_{\omega \to 0} G(j\omega) = 1 \angle 0$
- $\lim_{\omega \to \infty} G(j\omega) = \infty \angle 180$
Frequency domain analysis

\[ \omega \to \infty \]
**Gain Margin**

The gain margin is the reciprocal of the magnitude \(|G(j\omega)|\) at the frequency at which the phase angle is \(-180^\circ\) (Phased cross over frequency)

\[
K_g = \frac{1}{|G(j\omega)|} \quad \text{at} \angle G = -180^\circ
\]

**Phase Margin**

The phase margin is the amount of additional phase lag at the gain cross over frequency required to the verge of instability \(\gamma = 180^\circ + \phi\)
Phase and Gain Margin

Nyquist Diagram

Real Axis

Imaginary Axis

Phase Cross Over at (-0.5,0)

Gain Margin = 2

Unit Circle

Gain Cross Over

Phase Margin = 44.1

V. Sankaranarayanan

Frequency domain analysis
More Example

\[ G(s) = \frac{s + 1}{s + 100} \]
\[ G(j\omega) = \frac{j\omega + 1}{j\omega + 100} \]

- \( G(j0) = \frac{1}{100} \angle 0 \quad G(j\infty) = 1 \angle 0 \)
- \( \angle G = \tan^{-1} \omega - \tan^{-1} \frac{\omega}{100} > 0 \)
More Example

\[ G(s) = \frac{s + 100}{s + 1} \]

\[ G(j\omega) = \frac{j\omega + 100}{j\omega + 1} \]

- \( G(j0) = 100\angle0 \)
- \( G(j\infty) = 1\angle0 \)
- \( \angle G = \tan^{-1}\frac{\omega}{100} - \tan^{-1}\omega < 0 \)
- $G(s) = \frac{s + 5}{(s + 1)(s + 10)}$
- $G(j0) = 0.5 \angle 0 \quad G(j\infty) = 0 \angle -90^\circ$
Suppose $G(s) = \frac{1}{s(5s + 1)}$
\[ G(s) = \frac{e^{-s}}{2s + 1} \]

\[ G(j\omega) = \frac{e^{-j\omega}}{2j\omega + 1} \]

Magnitude = \( \frac{1}{\sqrt{4\omega^2 + 1}} \) Magnitude decreases with frequency

Phase = \(-\omega - \tan^{-1} 4\omega\) Phase is changes with frequency continuously.

Polar plot starts from 1∠0. Its spiral in nature.

V. Sankaranarayanan

Frequency domain analysis
More Example

- \( G(s) = \frac{s + 10}{(s + 1)(s^2 + 600s + 1000000)} \)
- \( G(j0) = 10^{-5} \angle 0 \quad G(j\infty) = 0 \angle -180^\circ \)
**System Type and Polar Plot**

\[ G(s) = \frac{(s + z_1)(s + z_2) \ldots (s + z_m)}{s^k(s + p_1)(s + p_2) \ldots (s + p_n)} \quad n > m \]

- Type 0 System:
  Starting point (\(\omega = 0\)) is finite and on real axis.
  Ending point (\(\omega = \infty\)) is at the origin.
**System Type and Polar Plot**

![Nyquist Diagram](image)

\[
G(s) = \frac{(s + z_1)(s + z_2) \cdots (s + z_m)}{s^n(p_1)(s + p_2) \cdots (s + p_n)} \quad n > m
\]

- **Type 1 System**: At \( \omega = 0 \) the magnitude is 0 and phase becomes \(-90^\circ\). At low frequency polar plot is asymptotic to a line parallel to imaginary axis. At \( \omega = \infty \) magnitude becomes zero.
\[ G(s) = \frac{(s + z_1)(s + z_2) \ldots (s + z_m)}{s^n(s + p_1)(s + p_2) \ldots (s + p_n)} \quad n > m \]

- Type 2 System: At \( \omega = 0 \) the magnitude is \( \infty \) but angle is \(-180^\circ\). So at lower frequency polar plot is asymptotic to a line parallel to negative real axis. At \( \omega = \infty \) magnitude is zero.