Control systems

Frequency domain analysis: Polar Plot

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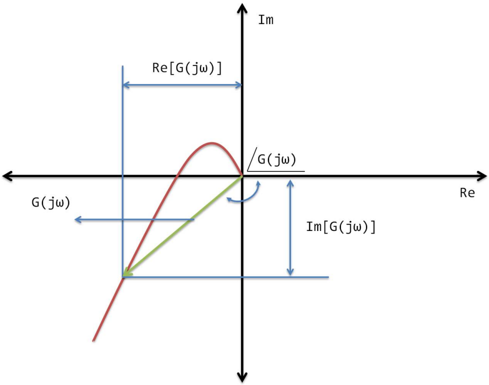
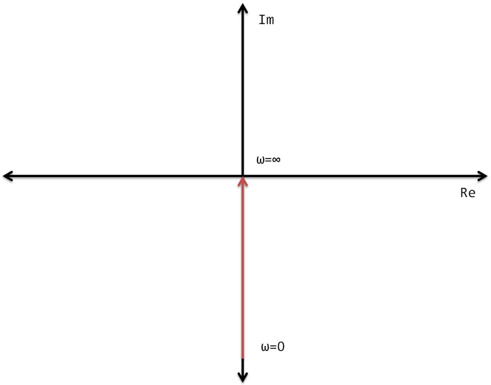
Outline

- It is a plot of the magnitude of $G(j\omega)$ versus the phase angle of $G(j\omega)$ on polar coordinates as ω is varied from zero to infinity
- In polar plots a positive (negative) phase angle is measured counterclockwise (clockwise) from the positive real axis



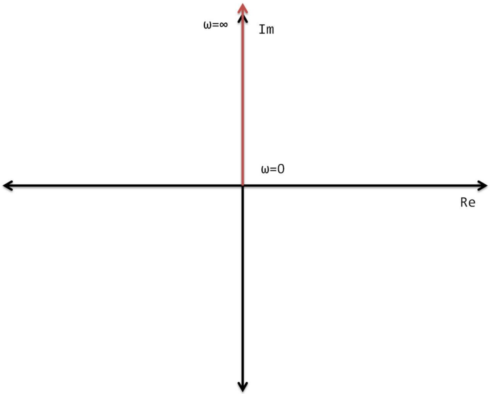
Integral and derivative factors

- $G(j\omega) = \frac{1}{j\omega} = -j\frac{1}{\omega} = \frac{1}{\omega}\angle{-90^o}$
- The Polar plot of $\frac{1}{j\omega}$ is the negative imaginary axis



Integral and derivative factors

- The Polar plot of $j\omega$ is the positive imaginary axis



First order factors

- $G(j\omega) = \frac{1}{1+j\omega T} = \frac{1}{\sqrt{1+\omega^2T^2}}\angle{-tan^{-1}\omega T}$
- $G(j0) = 1\angle 0^o$, $G(j\frac{1}{T}) = \frac{1}{\sqrt{2}}\angle 45^o$
- $\omega \longrightarrow \infty$, $|G(j\omega)| \longrightarrow 0$ and $\angle G(j\omega) \longrightarrow -90^o$

$$G(s) = X + jY$$

$$\frac{1}{1+j\omega T} = \frac{1-j\omega T}{1+\omega^2T^2}$$

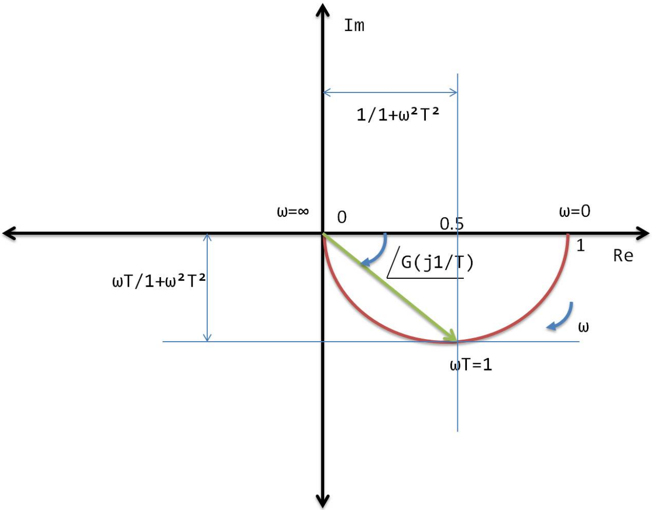
$$G(s) = \frac{1}{1+\omega^2T^2} - j\frac{wT}{1+\omega^2T^2}$$

$$X = \frac{1}{1+\omega^2T^2} \qquad Y = -\frac{wT}{1+\omega^2T^2}$$

$$\left(X - \frac{1}{2}\right)^2 + Y^2 = \left(\frac{1}{2}\right)^2$$

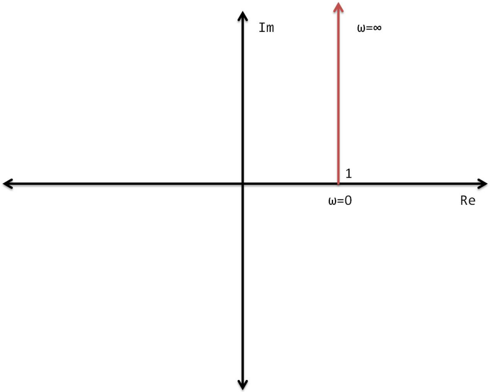
So the polar plot is a semi circle.

$$G(s) = \frac{1}{1 + j\omega T}$$



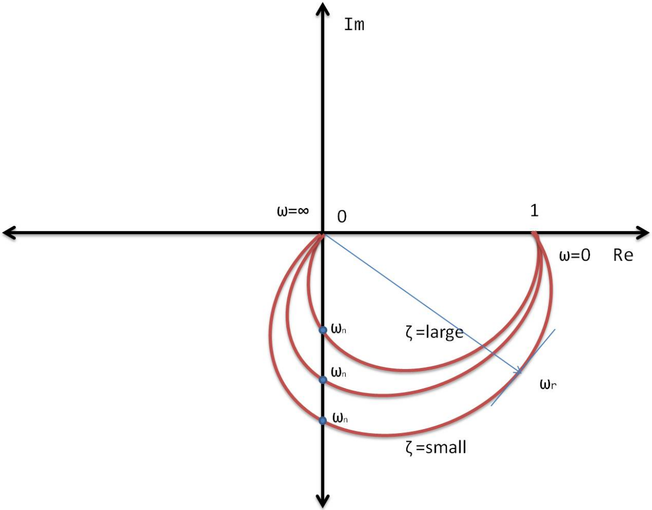
First order factors

- $G(j\omega) = 1 + j\omega T$
- It is simply the upper half of the straight line passing through point $(1, 0)$ in the complex plane and parallel to the imaginary axis



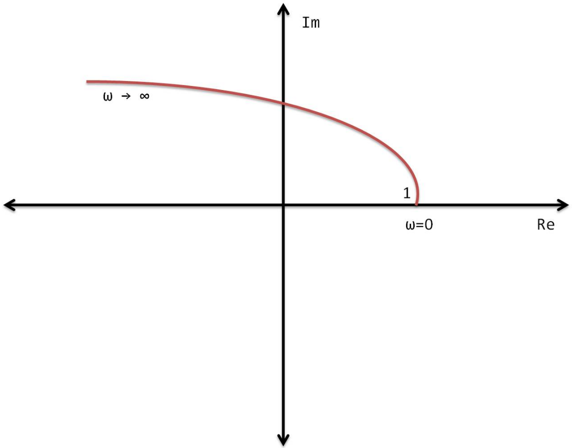
Quadratic factors

- $G(j\omega) = \frac{1}{1+2\zeta(j\frac{\omega}{\omega_n})+(j\frac{\omega}{\omega_n})^2}$
- For $\zeta > 0$
- $\lim_{\omega \to 0} G(j\omega) = 1\angle 0$
- $\lim_{\omega \to \infty} G(j\omega) = 0\angle -180$
- The polar plot of this sinusoidal transfer function starts at $1\angle 0$ and ends at $0\angle -180$ as ω increases from zero to infinity
- The high-frequency portion of $G(j\omega)$ is tangent to the negative real axis.
- For the underdamped case $\omega = \omega_n$, the phase angle is -90
- In the polar plot, the frequency point whose distance from the origin is maximum corresponds to the resonant frequency ω_r.


Im
ω=∞
0
1
ω=0
Re
ωn
ωn
ωn
ζ =large
ωr
ζ =small

Quadratic factors

- $G(j\omega) = 1 + 2\zeta(j\frac{\omega}{\omega_n}) + (j\frac{\omega}{\omega_n})^2$
- $= \left(1 - \frac{\omega^2}{\omega_n^2}\right) + j\left(\frac{2\zeta\omega}{\omega_n}\right)$
- $\lim_{\omega\to 0} G(j\omega) = 1\angle 0$
- $\lim_{\omega\to\infty} G(j\omega) = \infty\angle 180$


Im
ω → ∞
1
ω=0
Re

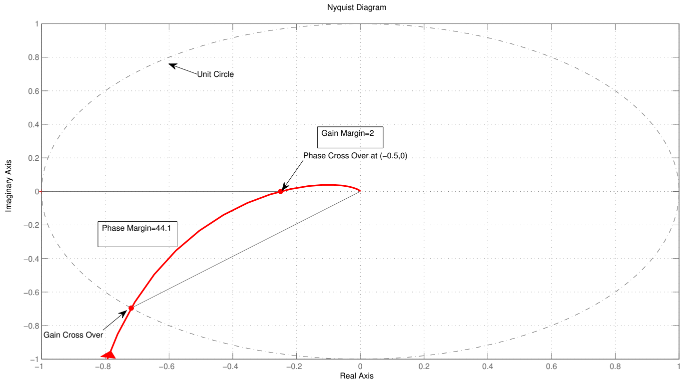
GAIN MARGIN

The gain margin is the reciprocal of the magnitude $|G(j\omega)|$ at the frequency at which the phase angle is -180^o(Phased cross over frequency)

$$K_g = \frac{1}{|G(j\omega)|} \;\; at\angle G = -180^o$$

PHASE MARGIN

The phase margin is the amount of additional phase lag at the gain cross over frequency required to the verge of instability $\gamma = 180^o + \phi$

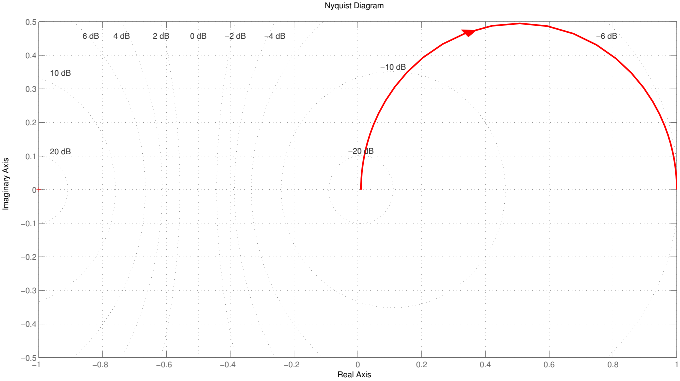


Nyquist Diagram
Unit Circle
Gain Margin=2
Phase Cross Over at (−0.5,0)
Phase Margin=44.1
Gain Cross Over
Imaginary Axis
Real Axis
1
0.8
0.6
0.4
0.2
0
−0.2
−0.4
−0.6
−0.8
−1
−1
−0.8
−0.6
−0.4
−0.2
0
0.2
0.4
0.6
0.8
1

More Example

$$G(s) = \frac{s+1}{s+100}$$
$$G(j\omega) = \frac{j\omega+1}{j\omega+100}$$

- $G(j0) = \frac{1}{100}\angle 0 \qquad G(j\infty) = 1\angle 0$
- $\angle G = \tan^{-1}\omega - \tan^{-1}\frac{\omega}{100} > 0$

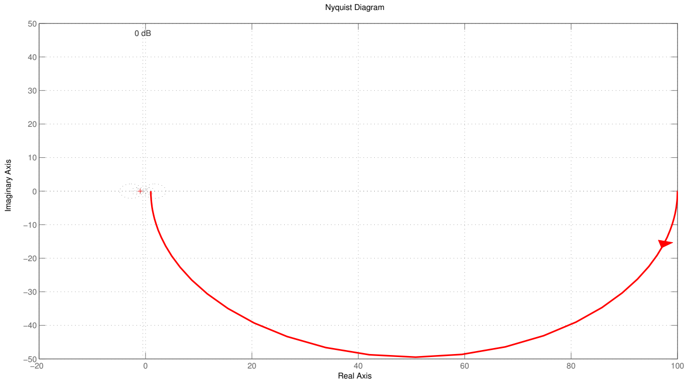


More Example

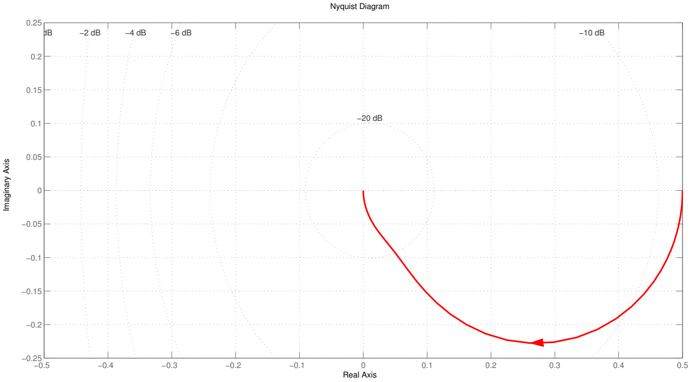
$$G(s) = \frac{s+100}{s+1}$$

$$G(j\omega) = \frac{j\omega+100}{j\omega+1}$$

- $G(j0) = 100\angle 0 \qquad G(j\infty) = 1\angle 0$
- $\angle G = \tan^{-1}\frac{\omega}{100} - \tan^{-1}\omega < 0$



- $G(s) = \dfrac{s+5}{(s+1)(s+10)}$
- $G(j0) = 0.5\angle 0 \qquad G(j\infty) = 0\angle -90^o$



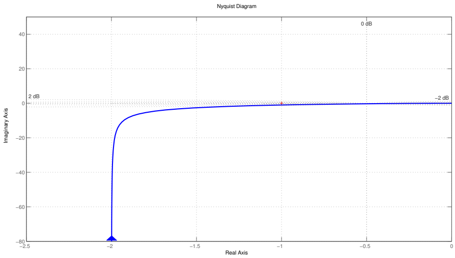
$$G(s) = \frac{1}{s(sT+1)}$$

$$G(j\omega) = \frac{1}{j\omega(j\omega T+1)} = -\frac{T}{1+\omega^2T^2} - j\frac{1}{\omega(1+\omega^2T^2)}$$

$$\lim_{\omega\to 0} G(j\omega) = -T - j\infty = \infty\angle - 90^o$$

$$\lim_{\omega\to\infty} G(j\omega) = 0 - j0 = 0\angle - 180^o$$

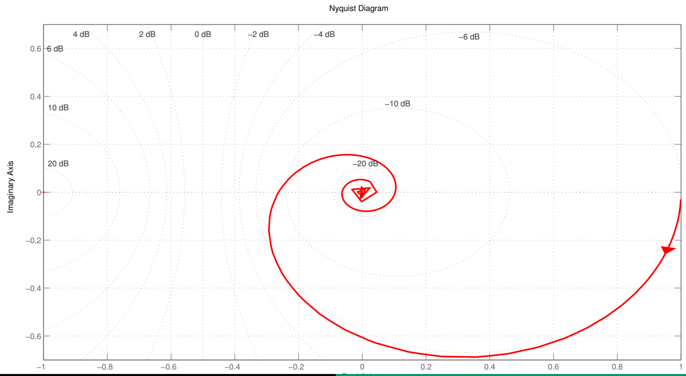
Suppose $G(s) = \frac{1}{s(5s+1)}$



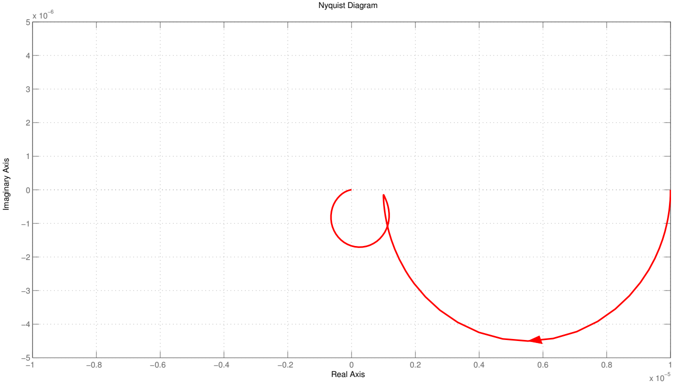
Example

$$G(s) = \frac{e^{-s}}{2s+1}$$
$$G(j\omega) = \frac{e^{-j\omega}}{2j\omega+1}$$

Magnitude= $\frac{1}{\sqrt{4\omega^2+1}}$ Magnitude decreases with frequency
Phase= $-\omega - \tan^{-1} 4\omega$ Phase is changes with frequency continously.
Polar plot starts from $1\angle 0$. Its spiral in nature.

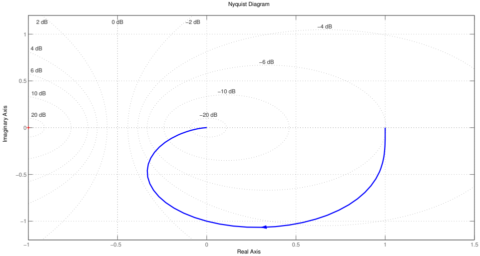


- $G(s) = \dfrac{s+10}{(s+1)(s^2+600s+1000000)}$
- $G(j0) = 10^{-5}\angle 0 \qquad G(j\infty) = 0\angle -180^o$



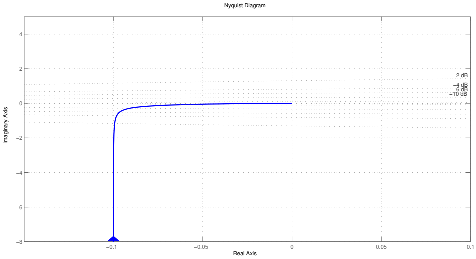
$$G(s) = \frac{(s+z_1)(s+z_2)\ldots\ldots(s+z_m)}{s^k(s+p_1)(s+p_2)\ldots\ldots(s+p_n)} n > m$$

- Type 0 System:
  Starting point($\omega = 0$) is finite and on real axis.
  Ending point($\omega = \infty$) is at the origin.



$$G(s) = \frac{(s+z_1)(s+z_2)\ldots\ldots(s+z_m)}{s^k(s+p_1)(s+p_2)\ldots\ldots(s+p_n)} n > m$$

- Type 1 System: At $\omega = 0$ the magnitude is 0 and phase become -90^o.At low frequency polar plot is asympototic to a line parallel to imaginary axis. At $\omega = \infty$ magnitude become zero.



$$G(s) = \frac{(s+z_1)(s+z_2)\ldots\ldots(s+z_m)}{s^k(s+p_1)(s+p_2)\ldots\ldots(s+p_n)} n > m$$

- Type 2 System: At $\omega = 0$ the magnitude is ∞ but angle is -180^o. So at lower frequency polar plot is asymptotic to a line parallel to negative real axis. At $\omega = \infty$ magnitude is zero.