Modern Control systems

Lecture-7 Observability

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Outline

Observability

The state space equations of the system is

$$\dot{x} = Ax + Bu \quad (1)$$
$$y = Cx + Du \quad (2)$$

where

$$x \in \mathbb{R}^n \; ; \; u \in \mathbb{R}^m \; ; \; y \in \mathbb{R}^p$$

$$\mathrm{A} \in \mathbb{R}^{n \times n} \qquad \mathrm{B} \in \mathbb{R}^{n \times m}$$
$$\mathrm{C} \in \mathbb{R}^{p \times n} \qquad \mathrm{D} \in \mathbb{R}^{p \times m}$$

Definition

- A system of the form (1) is said to be observable if the initial state x(0) can be computed from the knowledge of the input u(t) and output y(t) for a given period of time T
- Observability is concerned with what can be said about initial state when given measurements of the plant.

Observability

Definition

The state equation is said to be observable if for any unknown initiaal state x(0), there exists a finite $t_1 > 0$ such that the knowledge of the input u and the input y over $[0.t_1]$ suffices to determine uniquely the initial state $x(0)$. Otherwise, the equation is said to be unobservable

Observability-Rank condition

The system of the form (1) is observable if and only if the rank of $\mathcal{O} = \begin{pmatrix} C \\ CA \\ CA^2 \\ . \\ . \\ CA^{n-1} \end{pmatrix}$ is n (order of the system)

Observability

Rank condition

- $y = Cx + Du$
- $\dot{y} = CAx + CBu + D\dot{u}$
- $\ddot{y} = CA^2x + CABu + CB\dot{u} + D\ddot{u}$ ......
- $$\begin{pmatrix} y \\ \dot{y} \\ \ddot{y} \\ . \\ . \\ y^{n-1} \end{pmatrix} = \begin{pmatrix} C \\ CA \\ CA^2 \\ . \\ . \\ CA^{n-1} \end{pmatrix} xt + \begin{pmatrix} D & 0 & 0 & 0 & . & . \\ CB & D & 0 & 0 & . & . \\ CAB & CB & D & 0 & . & . \\ . & . & . & . & . & . \end{pmatrix} \begin{pmatrix} u \\ \dot{u} \\ \ddot{u} \\ . \\ . \\ u^{n-1} \end{pmatrix}$$
- $y = [\mathcal{O}]x(t) + [H]u(t)$
  $x(t) = \mathcal{O}^{-1}[y(t) - [H]u(t)]$
- $[\mathcal{O}]^{-1}$ should exist $\rightarrow$ Rank of $\mathcal{O}$ should be n

Observability

Example-1

Consider the state and output equations,

$$\dot{x}_1 = -x_1 + x_2$$
$$\dot{x}_2 = -3x_2 + u$$
$$y = x_1 + x_2$$

- Let us try to check for Observability of this system by transforming these equations into different forms.
- Representing these equations in matrix form , we get

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ 0 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$
$$y = \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Observability

Example-1

- We know that the Observability matrix for a system having two state variables is given by

$$\mathcal{V} = \begin{bmatrix} \boldsymbol{C} \\ \boldsymbol{CA} \end{bmatrix}_{2\times 2}$$

$$\therefore \mathcal{V} = \begin{bmatrix} 1 & 1 \\ -1 & -2 \end{bmatrix}$$

- Rank of matrix $\mathcal{V} = 2$
- Therefore, the system is Observable.

Observability

Example-1

- Now, let us check the Observability of the system by transforming it into diagonal canonical form.
- The transformation matrix $\boldsymbol{P}$ is the modal matrix formed by the eigenvectors corresponding to the eigenvalues of matrix $\boldsymbol{A}$.
- Matrix $\boldsymbol{P} = \begin{bmatrix} 1 & 1 \\ 0 & -2 \end{bmatrix}$
- Transformed matrices $\boldsymbol{A_z}$,$\boldsymbol{B_z}$ and $\boldsymbol{C_z}$ are obtained using relations $\boldsymbol{A_z} = \boldsymbol{P^{-1}AP}, \boldsymbol{B_z} = \boldsymbol{P^{-1}B}, \boldsymbol{C_z} = \boldsymbol{CP}$

$$\boldsymbol{A_z} = \begin{bmatrix} -1 & 0 \\ 0 & -3 \end{bmatrix} \boldsymbol{B_z} = \begin{bmatrix} 1/2 \\ -1/2 \end{bmatrix} \boldsymbol{C_z} = \begin{bmatrix} 1 & -1 \end{bmatrix}$$

Observability

Example-1

- Representing them in matrix form , we get

$$\begin{aligned} \begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \end{bmatrix} &= \begin{bmatrix} -1 & 0 \\ 0 & -3 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} + \begin{bmatrix} 1/2 \\ -1/2 \end{bmatrix} u \\ y &= \begin{bmatrix} 1 & -1 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} \end{aligned}$$

- Observability matrix, $\boldsymbol{\mathcal{V}} = \begin{bmatrix} 1 & -1 \\ -1 & 3 \end{bmatrix}$
- Rank of the matrix $\boldsymbol{\mathcal{V}} = 2$.
- Therefore, the system is Observable.

Observability

Example-1

- Now, let us check the observability of the system by transforming it into Canonical Form-II using Similarity Transformation.
- The transformation matrix $\boldsymbol{P}$ is obtained by Conventional method, explained in Lecture-5.
- Matrix $\boldsymbol{P} = \begin{bmatrix} 1 & -2 \\ -1 & 3 \end{bmatrix}$
- Transformed matrices $\boldsymbol{A_z}$, $\boldsymbol{B_z}$ and $\boldsymbol{C_z}$ are obtained using relations $\boldsymbol{A_z} = \boldsymbol{P^{-1}AP}, \boldsymbol{B_z} = \boldsymbol{P^{-1}B}$ and $\boldsymbol{C_z} = \boldsymbol{CP}$

$$\boldsymbol{A_z} = \begin{bmatrix} 0 & -3 \\ 1 & -4 \end{bmatrix} \boldsymbol{B_z} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \boldsymbol{C_z} = \begin{bmatrix} 0 & 1 \end{bmatrix}$$

Observability

Example-1

- Representing them in matrix form , we get

$$\begin{bmatrix}\dot{z}_1\\ \dot{z}_2\end{bmatrix} = \begin{bmatrix}0 & -3\\ 1 & -4\end{bmatrix}\begin{bmatrix}z_1\\ z_2\end{bmatrix} + \begin{bmatrix}2\\ 1\end{bmatrix}u$$

$$y = \begin{bmatrix}0 & 1\end{bmatrix}\begin{bmatrix}z_1\\ z_2\end{bmatrix}$$

- Controllability matrix, $\boldsymbol{\mathcal{V}} = \begin{bmatrix}0 & 1\\ 1 & -4\end{bmatrix}$
- Rank of the matrix $\boldsymbol{\mathcal{V}} = 2$.
- Therefore, the system is Observable.

Observability

Example-2

Consider the state and output equations,

$$\dot{x}_1 = 2x_2$$
$$\dot{x}_2 = -4x_1 - 6x_2 + u$$
$$y = x_1 + x_2$$

- Let us try to check for Observability of this system by transforming these equations into different forms.
- Representing these equations in matrix form , we get

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 2 \\ -4 & -6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$
$$y = \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Observability

Example-2

- We know that the Observability matrix for a system having two state variables is given by

$$\boldsymbol{\mathcal{V}} = \begin{bmatrix} \boldsymbol{C} \\ \boldsymbol{CA} \end{bmatrix}_{2\times 2}$$

$$\therefore \boldsymbol{\mathcal{V}} = \begin{bmatrix} 1 & 1 \\ -4 & -4 \end{bmatrix}$$

- Rank of matrix $\boldsymbol{\mathcal{V}} = 1$, less than 2.
- Therefore, the system is not Observable.

Observability

Example-2

- Now, let us check the Observability of the system by transforming it into diagonal canonical form.
- The transformation matrix $\boldsymbol{P}$ is the modal matrix formed by the eigenvectors corresponding to the eigenvalues of matrix $\boldsymbol{A}$.
- Matrix $\boldsymbol{P} = \begin{bmatrix} 1 & 1 \\ -1 & -2 \end{bmatrix}$
- Transformed matrices $\boldsymbol{A_z}$,and $\boldsymbol{C_z}$ are obtained using relations $\boldsymbol{A_z} = \boldsymbol{P^{-1}AP}, \boldsymbol{C_z} = \boldsymbol{CP}$

$$\boldsymbol{A_z} = \begin{bmatrix} -2 & 0 \\ 0 & -4 \end{bmatrix} \boldsymbol{C_z} = \begin{bmatrix} 0 & -1 \end{bmatrix}$$

Observability

Example-2

- Representing them in matrix form , we get

$$\begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \end{bmatrix} = \begin{bmatrix} -2 & 0 \\ 0 & -4 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} + \begin{bmatrix} 1 \\ -1 \end{bmatrix} u$$

$$y = \begin{bmatrix} 0 & -1 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}$$

- Observability matrix, $\boldsymbol{\mathcal{V}} = \begin{bmatrix} 0 & -1 \\ 0 & 4 \end{bmatrix}$
- Rank of the matrix $\boldsymbol{\mathcal{V}} = 1$.
- Therefore, the system is not Observable.

Observability

Example-2

- Now, let us check the observability of the system by transforming it into Canonical Form-II using Similarity Transformation.
- The transformation matrix $\boldsymbol{P}$ is obtained by Conventional method, explained in Lecture-5.
- Matrix $\boldsymbol{P} = \begin{bmatrix} 1 & 0 \\ 0 & -4 \end{bmatrix}$
- Transformed matrices $\boldsymbol{A_z}$, $\boldsymbol{B_z}$ and $\boldsymbol{C_z}$ are obtained using relations $\boldsymbol{A_z} = \boldsymbol{P^{-1}AP}, \boldsymbol{B_z} = \boldsymbol{P^{-1}B}$ and $\boldsymbol{C_z} = \boldsymbol{CP}$

$$\boldsymbol{A_z} = \begin{bmatrix} 0 & -8 \\ 1 & -6 \end{bmatrix} \boldsymbol{B_z} = \begin{bmatrix} 0 \\ -1/4 \end{bmatrix} \boldsymbol{C_z} = \begin{bmatrix} 1 \\ -4 \end{bmatrix}$$

Observability

Example-2

- Representing them in matrix form , we get

$$\begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \end{bmatrix} = \begin{bmatrix} 0 & -8 \\ 1 & -6 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} + \begin{bmatrix} 0 \\ -1/4 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & -4 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}$$

- Controllability matrix, $\boldsymbol{\mathcal{V}} = \begin{bmatrix} 1 & -4 \\ 1 & -4 \end{bmatrix}$
- Rank of the matrix $\boldsymbol{\mathcal{V}} = 1$.
- Therefore, the system is not Observable.

Assignment Questions on Controllability & Observability

1. State space representation of the system is given by

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} -1 & -2 & -2 \\ 0 & -1 & 1 \\ 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} u; \qquad y = \begin{bmatrix} 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \tag{3}$$

   Check whether the system is completely state controllable?

2. Check the controllability of the system (3) by transforming it into diagonal canonical form.
3. Check the controllability of the system (3) by transforming it into Canonical Form-I using Similarity Transformation.
4. State space representation of the system is given by

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 3 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}; \qquad y = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \tag{4}$$

   Check whether the system is completely state controllable?

5. Check the controllability of the system (4) by transforming it into diagonal canonical form.
6. Check the controllability of the system (4) by transforming it into Canonical Form-I using Similarity Transformation.
7. State space representation of the system is given by

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u; \qquad y = \begin{bmatrix} 20 & 9 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \tag{5}$$

   Check whether the system is completely state observable?

8. Check the observability of the system (5) by transforming it into diagonal canonical form.

9. Check the observability of the system (5) by transforming it into Canonical Form-II using Similarity Transformation.
10. State space representation of the system is given by

$$\begin{bmatrix} \dot{x_1} \\ \dot{x_2} \\ \dot{x_3} \end{bmatrix} = \begin{bmatrix} 1 & 1 & 2 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}; \qquad y = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \tag{6}$$

    Check whether the system is completely state observable?
11. Check the observability of the system (6) by transforming it into diagonal canonical form.
12. Check the observability of the system (6) by transforming it into Canonical Form-II using Similarity Transformation.