Modern Control systems

Lecture-2 Mathematical Modelling

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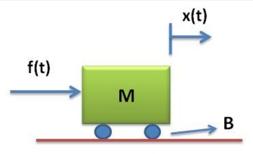
Outline

Outline

Example-1

A mechanical system



$$M\frac{d^2x}{dt^2} + B\frac{dx}{dt} = f(t)$$

Let $x_1 = x$; $x_2 = \frac{dx}{dt}$

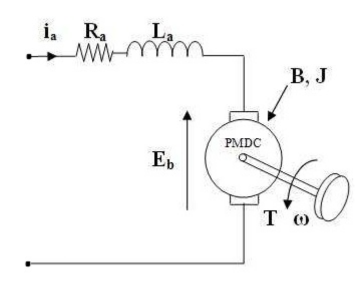
$$\dot{x_1} = x_2$$
$$\dot{x_2} = \frac{f(t)}{M} - \frac{B}{M}x_2$$

$$\begin{pmatrix}\dot{x_1}\\ \dot{x_2}\end{pmatrix} = \begin{pmatrix}0 & 1\\ 0 & \frac{-B}{M}\end{pmatrix} + \begin{pmatrix}0\\ \frac{1}{M}\end{pmatrix}u(t)$$

$$y = \begin{pmatrix}1 & 0\end{pmatrix}\begin{pmatrix}x_1\\ x_2\end{pmatrix} + (0)u(t)$$

Example-2

Electro-Mechanical system-DC MOTOR



Governing Equations:

$E_b + R_a i_a + L_a \frac{di_a}{dt} = V$

$J\frac{d\omega}{dt} + B\omega + T_l = T$

$E_b = k_b\omega \qquad (\omega = \frac{d\theta}{dt} = \dot{\theta})$

$T = k_b i_a$

Example-2

Continuation of Example-2....

Let, $x_1 = \omega, x_2 = i_a$ and let, 'y' be the output of the system.

$$\dot{x_1} = \dot{\omega}, \qquad \dot{x_2} = \dot{i_a}$$

$$\dot{\omega} = \frac{-B}{J}\omega + \frac{k_b}{J}i_a$$

$$\dot{i_a} = \frac{-k_b}{L_a}\omega - \frac{R_a}{L_a}i_a + \frac{1}{L_a}V$$

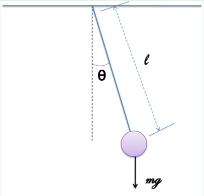
$$y = x_1(t) = \omega(t)$$

$$\therefore \begin{bmatrix} \dot{x_1} \\ \dot{x_2} \end{bmatrix} = \begin{bmatrix} \frac{-B}{J} & \frac{k_b}{J} \\ \frac{-k_b}{L_a} & \frac{-Ra}{L_a} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{L_a} \end{bmatrix} V$$

$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + [0]V$$

Example-3

Mechanical System



Governing Equation:

$$mgl^2\frac{d^2\theta}{dt^2} + mgl\sin(\theta) + kl\dot{\theta} = T$$

This is a nonlinear system, linearized by assuming

$$\sin(\theta) \approx \theta$$

The governing equation becomes:

$$mgl^2\frac{d^2\theta}{dt^2} + mgl\theta + kl\dot{\theta} = T$$

Example-3

Continuation of Example-3...

Let, 'y' be the output of the system.

Let, $x_1 = \theta$,
$x_2 = \dot{\theta} = \dot{x_1}$,

$\dot{x_2} = \ddot{\theta}$,

$\ddot{\theta} = \frac{-1}{l}\theta - \frac{k}{mgl}\dot{\theta} + \frac{1}{mgl^2}$

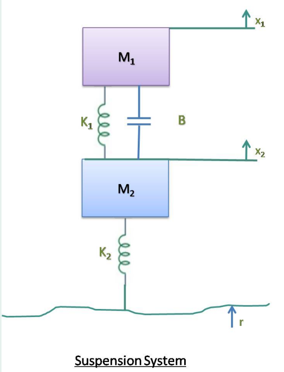
$y(t) = \theta(t) = x_1(t)$

$$\therefore \begin{bmatrix} \dot{x_1} \\ \dot{x_2} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ \frac{1}{l} & \frac{k}{mgl} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{mgl^2} \end{bmatrix} T$$

$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + [0]T$$

Example-4

Automobile Suspension System.



Governing Equations:

$$M_1 \frac{d^2 x_1}{dt^2} = -k_1(x_1 - x_2) - B(\dot{x_1} - \dot{x_2}),$$
$$M_2 \frac{d^2 x_2}{dt^2} = k_1(x_1 - x_2) + B(\dot{x_1} - \dot{x_2}) + k_2(r - x_2)$$

Example-4

Continuation of Example-4...

Let, 'y' be the output of the system and 'r' be the displacement input.

$$\frac{d^2x_1}{dt^2} = \frac{-k_1}{M_1}(x_1 - x_2) - \frac{B}{M_1}(\dot{x_1} - \dot{x_2})$$

$$\frac{d^2x_2}{dt^2} = \frac{k_1}{M_2}(x_1 - x_2) + \frac{B}{M_2}(\dot{x_1} - \dot{x_2}) + \frac{k_2}{M_2}(r - x_2)$$

Let,

$$z_1 = x_1,$$

$$z_2 = x_2,$$

$$z_3 = \dot{x_1} = \frac{dx_1}{dt},$$

$$z_4 = \dot{x_2} = \frac{dx_2}{dt}$$

$$y = x_1 = z_1$$

$$\dot{z_3} = \frac{-k_1}{M_1}(z_1 - z_2) - \frac{B}{M_1}(z_3 - z_4)$$

$$\dot{z_4} = \frac{k_1}{M_2}(z_1 - z_2) + \frac{B}{M_2}(z_3 - z_4) + \frac{k_2}{M_2}(r - z_2)$$

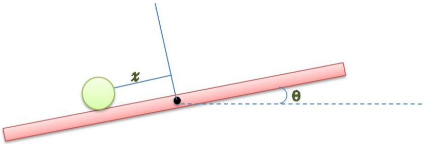
Example-4

Continuation of Example-4

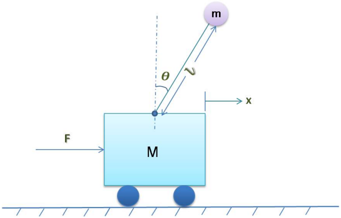
$$\therefore \begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \\ \dot{z}_3 \\ \dot{z}_4 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \frac{-k_1}{M_1} & \frac{k_1}{M_1} & \frac{-B}{M_1} & \frac{B}{M_1} \\ \frac{k_1}{M_2} & \frac{-(k_1+k_2)}{M_2} & \frac{B}{M_2} & \frac{-B}{M_2} \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ \frac{k_2}{M_2} \end{bmatrix} r$$

$$y = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{bmatrix} + [0] r$$

Ball on a beam



Inverted pendulum on a cart



Equations of motion

- $F = M\ddot{x} + m\frac{d^2}{dt^2}(x + l\sin\theta)$
- $F = (M+m)\ddot{x} + ml\dot{\theta}^2\sin\theta - ml\ddot{\theta}\cos\theta$
- $(I + ml^2)\ddot{\theta} = mgl\sin\theta - m\ddot{x}l\cos\theta$