Control systems

Lecture-2: Mathematical Modelling

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Outline

Representations

Dynamical system

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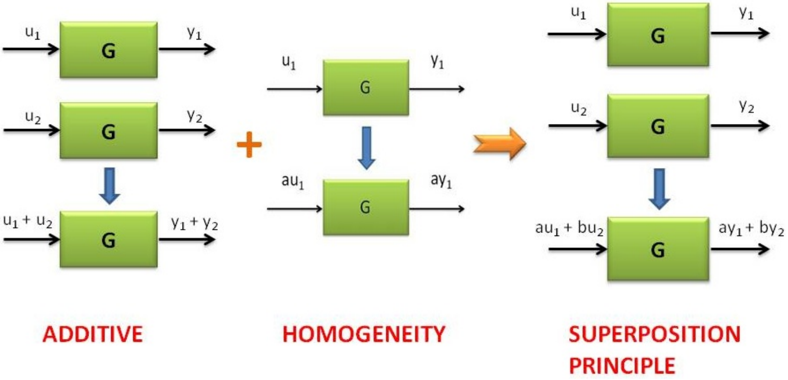
Linear system

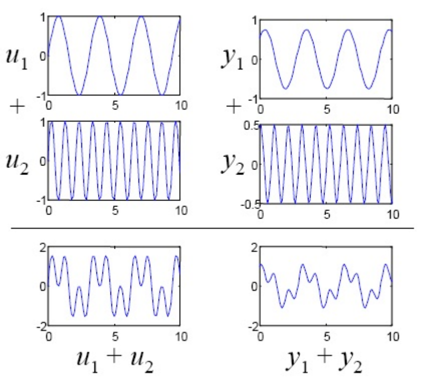
Definition

A system is said to be linear in terms of the system input $u(t)$ and the system output $y(t)$ if it satisfies the additive and homogeneity property which implies that it satisfies the superposition principle.

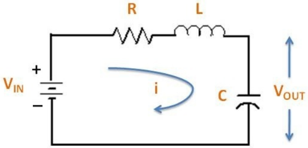


Linear system

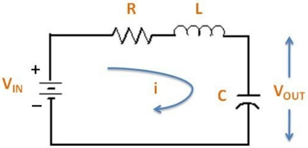
Additive property



Mathematical modelling

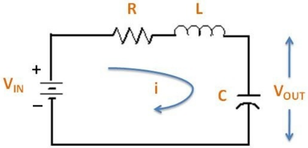


Mathematical modelling



Is this linear system ?

Mathematical modelling



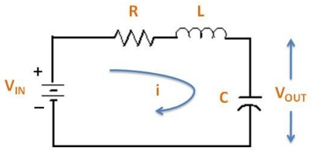
Is this linear system ?

- Mathematical relation

$$V_{in} = Ri + L\frac{di}{dt} + \frac{1}{C}\int idt$$

$$V_{out} = \frac{1}{C}\int idt$$

Mathematical modelling



Is this linear system ?

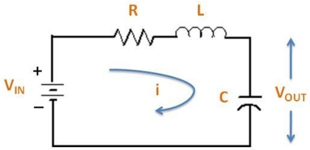
- Mathematical relation

$$V_{in} = Ri + L\frac{di}{dt} + \frac{1}{C}\int idt$$

$$V_{out} = \frac{1}{C}\int idt$$

- How to find $V_{out}(t)$

Why we need to find $V_{out}(t)$



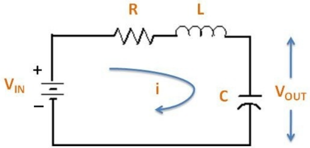
Modelling

- The behaviour of the output
- Passive design
  - What values of V_{in}, R, L and C for particular output value at particular time

$$V_{out}(2.8) = 3.9V$$

- Active design
  - How to get the desired output value by changing the input ?
  - Controller

How to obtain $V_{out}(t)$



Let V$_{in}$=10V,R=3Ω,L=1H,C=0.5F

$$V_{in} = Ri + L\frac{di}{dt} + \frac{1}{C}\int idt$$

On Differentiating, we get

$$R\frac{di}{dt} + L\frac{di^2}{dt^2} + \frac{i}{C} = 0$$

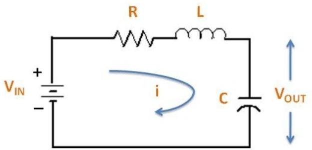
$$R\dot{i} + L\ddot{i} + \frac{i}{C} = 0$$

Sub for R,L,C values,we get

$$z^2 + 3z + 2 = 0$$

$$(z+1)(z+2) = 0$$

$$i(t) = c_1e^{-t} + c_2e^{-2t}$$



$$
\begin{aligned}
V_{out}(t) &= \frac{1}{0.5}\int i(t)dt \\
&= 2(-c_1e^{-t} - \frac{c_2}{2}e^{-2t}) + K
\end{aligned}
$$

When $V_{out}(\infty) = 10$; K=10

Thus,

$$V_{out} = 2(-c_1e^{-t} - \frac{c_2}{2}e^{-2t}) + 10$$

When $V_{out}(0) = 0 \rightarrow 2c_1 + c_2 = 10$

$i(0) = 0 \rightarrow c_1 + c_2 = 0$

$c_1 = 10; c_2 = -10$

$$V_{out} = 10 + 10e^{-2t} - 20e^{-t}$$

Is there an another way ?

Laplace Transform

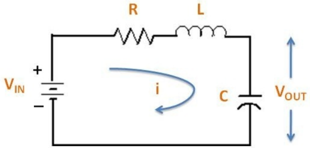
Definition

$$\mathcal{L}\{f(t)\} = F(s) = \int_0^\infty f(t)e^{-st}dt$$

Definition

- $$\mathcal{L}\{f^{'}\} = s\mathcal{L}\{f\} - f(0)$$
- $$\mathcal{L}\{f^{''}\} = s^2\mathcal{L}\{f\} - sf(0) - f^{'}(0)$$

How to compute $V_{out}(t)$? using Laplace Transform



$$V_{in} = Ri + L\frac{di}{dt} + \frac{1}{C}\int dt$$

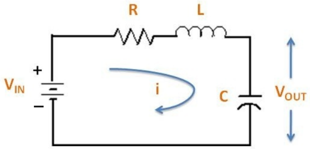
$$V_{out} = \frac{1}{C}\int dt$$

Taking Laplace transform

$$V_{in}(s) = i(s)\left[R + sL + \frac{1}{sC}\right]$$

$$V_{out}(s) = \frac{1}{sC}i(s)$$

$$V_{out}(s) = \frac{1}{sC}\left[\frac{V_{in}(s)}{(R + sL + \frac{1}{sC})}\right]$$



Let V_{in}=10V,R=3Ω,L=1H,C=0.5F

$$V_{out}(s) = V_{in}(s)\frac{2}{(s+1)(s+2)}$$

$$V_{out}(s) = \frac{20}{s(s+1)(s+2)}$$

Taking Laplace inverse transform,

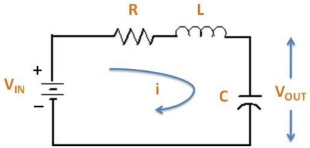
$$V_{out}(t) = 10 - 20e^{-t} + 10e^{-2t}$$

Definitions

Transfer function

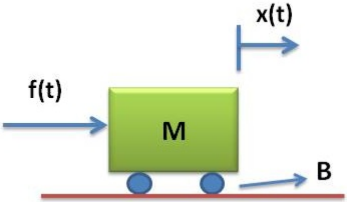
Ratio between the Laplace transform of input and output, that is,

$$\frac{Y(s)}{U(s)}$$



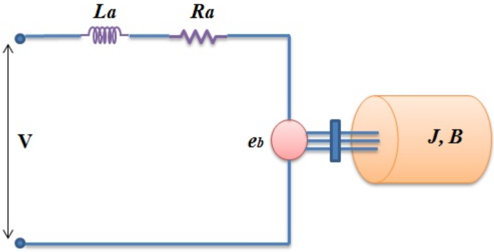
$$\frac{V_{out}(s)}{V_{in}(s)} = \frac{1}{sC}\left[\frac{1}{\left(R + sL + \frac{1}{sC}\right)}\right]$$

Mechanical system

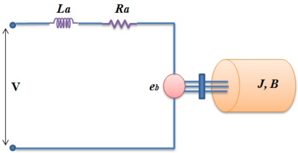


- $$M\frac{d^2x}{dt^2} + B\frac{dx}{dt} + Kx = F(t)$$
- Taking Laplace transform
$$Ms^2x(s) + Bsx(s) + Kx(s) = F(s)$$
- Transfer function
$$\frac{x(s)}{F(s)} = \frac{1}{Ms^2 + Bs + k}$$

PMDC Motor



- $$\begin{aligned} V &= R_a i + L_a \frac{di}{dt} + e_b \\ e_b &= k_b \omega \\ T &= k_t i \\ T &= J\frac{d\omega}{dt} + B\omega + T_l \end{aligned}$$



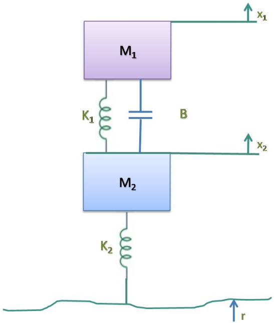
- Taking Laplace transform

$$\begin{aligned}
V(s) &= R_a i(s) + sL_a i(s) + e_b(s) \\
e_b(s) &= k_b \omega(s) \\
T(s) &= k_t i(s) \\
T(s) &= Js\omega(s) + B\omega(s)
\end{aligned}$$

- The transfer function

$$\frac{\omega(s)}{V(s)} = \frac{k_b}{JL_a s^2 + s(BL_a + JR_a) + (BR_a + k_b^2)}$$

Suspension system

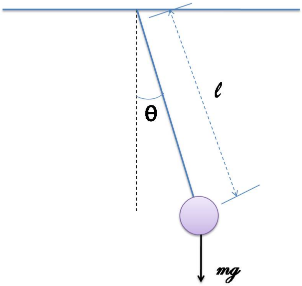


Suspension System

$$m_2 \frac{d^2 y}{dt^2} + k_s(y - x) + b\frac{d}{dt}(y - x) = 0$$

$$m_1 \frac{d^2 x}{dt} + b\frac{d}{dt}(x - y) + k_s(x - y) = k_w(r - x)$$

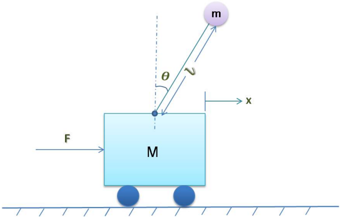
Pendulum



- $mgl^2\ddot{\theta} + mgl\sin\theta + kl\dot{\theta} = T$
- Linearization $\sin\theta \approx \theta$
- $mgl^2\ddot{\theta} + mgl\theta + kl\dot{\theta} = T$
- Calculate the transfer function

$$G(s) = \frac{\theta(s)}{T(s)}$$

Inverted pendulum on a cart

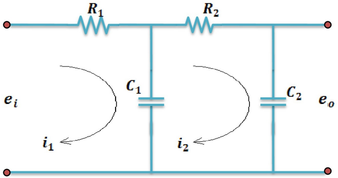


Equations of motion

- $F = M\ddot{x} + m\frac{d^2}{dt^2}(x + l\sin\theta)$
- $F = (M+m)\ddot{x} + ml\dot{\theta}^2\sin\theta - ml\ddot{\theta}\cos\theta$
- $(I + ml^2)\ddot{\theta} = mgl\sin\theta - m\ddot{x}l\cos\theta$

Electrical Systems

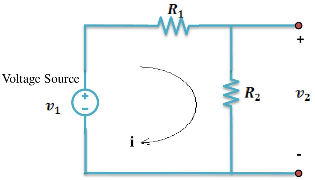
1. Find the transfer function of the system given below by taking input output as given



- Input e_i
- Output e_0

Electrical Systems

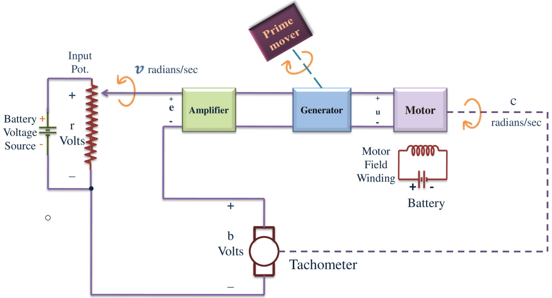
2. Find the transfer function of the system given below by taking input output as given



- Input v_1
- Output v_2

Electro-Mechanical Systems

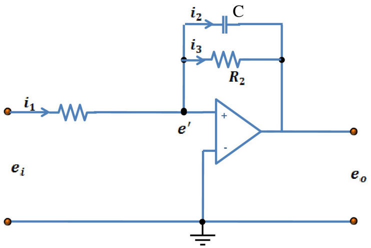
3. Find the transfer function of the system given below by taking input output as given



- Input Battery voltage
- Output Motor speed

Operational Amplifier Circuit

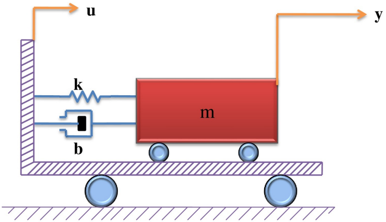
4. Find the transfer function of the system given below by taking input output as given



- Input e_i
- Output e_0

Spring-mass System on a Cart

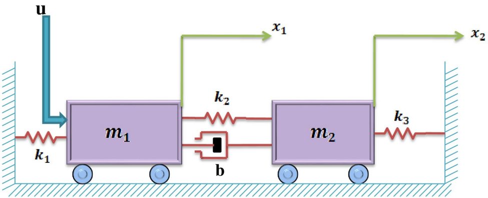
5. Find the transfer function of the system given below by taking input output as given



- Input u
- Output y

Cascaded Spring Mass System

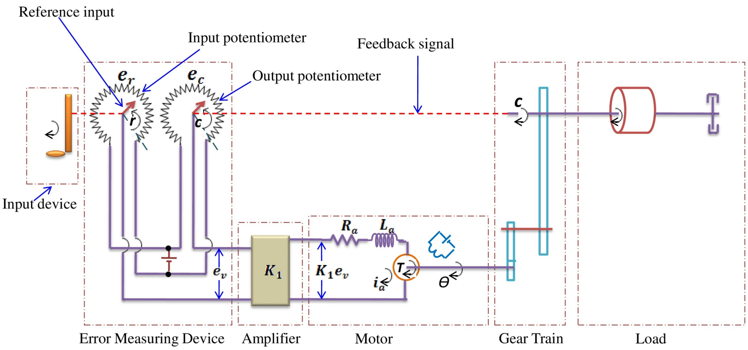
6. Find the transfer function of the system given below by taking input output as given



- Input u
- Output can be x_1 or x_2

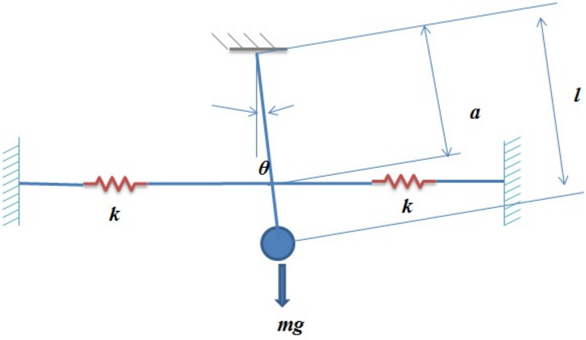
Mechanical System

7. Consider the servo system with a dc motor given in the figure. A pair of potentiometer act as a error measuring device. They convert input output position into electrical signals. Input is r and output is c. The error signal is $e = r - c$. The error voltage is $e_v = e_r - e_c$. And $e_r = K_0 r, e_c = K_0 c$. For the constant field current motor torque is $T = K_2 i_a$. Determine the transfer function of the system.



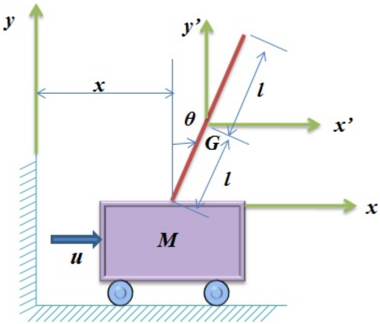
Spring-Loaded Pendulum

8. Consider the spring-loaded pendulum system.Assume that the spring force acting on the pendulum is zero when the pendulum is vertical, or $\theta = 0$. Assume also that the friction involved is negligible and the angle of oscillation θ is small. Obtain a mathematical model of the system.



Inverted Pendulum System

9. Consider the inverted-pendulum.Assume that the mass of the inverted pendulum is m and is evenly distributed along the length of the rod. (The center of gravity of the pendulum is located at the center of the rod.) Assuming that θ is small, derive mathematical models for the system in the forms of differential equations.



Servomotor System

10. Consider the system. An armature-controlled dc servomotor drives a load consisting of the moment of inertia J_L. The torque developed by the motor is T. The moment of inertia of the motor rotor is J_m. The angular displacements of the motor rotor and the load element are θ_m and θ, respectively. The gear ratio is n = θ / θ_m. Obtain the transfer function $\theta(s)$ / $E_i(s)$.