Modern Control systems

Lecture-6 Controllability

V. Sankaranarayanan

Outline

Controllability

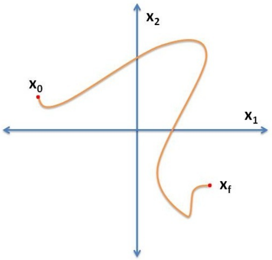
The state space equations of the system is

$$\dot{x} = \boldsymbol{A}x + \boldsymbol{B}u \quad (1)$$
$$y = \boldsymbol{C}x + \boldsymbol{D}u \quad (2)$$

where

$$x \in \mathbb{R}^n \; ; \; u \in \mathbb{R}^m \; ; \; y \in \mathbb{R}^p$$

$\boldsymbol{A} \in \mathbb{R}^{n \times n}$ $\quad$ $\boldsymbol{B} \in \mathbb{R}^{n \times m}$
$\boldsymbol{C} \in \mathbb{R}^{p \times n}$ $\quad$ $\boldsymbol{D} \in \mathbb{R}^{p \times m}$



Definition

A linear system, described above by state space equations (1) and is said to be controllable, if for any initial state $x(0) = x_0$ and any final state $x(T) = x_f$, there exists an unconstrained control input $u(t)$,$0 \leq t \leq T$ that transfers the system from x_0 to x_f in a finite time 'T'. Otherwise the system is said to be uncontrollable.

Controllability

Definition

- A state x_0 is said to be Controllable, if there exists a finite time interval $[0, T]$ and an input $u(t), t \in [0, T]$ such that $x(T) = 0$
- If all sates are controllable, then the system is said to be Completely Controllable

Controllability - Rank condition

The system of the form (1) is said to be controllable if and only if the rank of
$\boldsymbol{\mathcal{C}}=\begin{pmatrix}\boldsymbol{B} & \boldsymbol{AB} & \boldsymbol{A^2B} & . & . & . & \boldsymbol{A^{n-1}B}\end{pmatrix} = n$

Proof

- $x(t) = e^{\boldsymbol{A}t}\boldsymbol{x}(0) + \int_0^t e^{\boldsymbol{A}(t-\tau)}\boldsymbol{B}u(\tau)d\tau$
- Since the controllability depends only on input state variables.
  $x(t) = \int_0^t e^{\boldsymbol{A}(t-\tau)}\boldsymbol{B}u(\tau)d\tau$
  $x(t) = \int_0^t e^{\boldsymbol{A}t}\boldsymbol{B}u(t-\tau)d\tau$
- $e^{\boldsymbol{A}t} = I + \boldsymbol{A}t + \frac{(\boldsymbol{A}t)^2}{2!} + \frac{(\boldsymbol{A}t)^3}{3!} + ...$
  $e^{\boldsymbol{A}t} = \sum_{i=0}^{n-1} \boldsymbol{A}^i\alpha_i(t)$

Controllability

Controllability - Rank condition

- $x(t) = \sum_{i=0}^{n-1} \boldsymbol{A}^i \boldsymbol{B} \beta_i(t)$

  where $\beta_i(t) = \int_0^t \alpha_i(t) u(t-\tau) d\tau$

- $$x(t) = \begin{pmatrix} \boldsymbol{B} & \boldsymbol{AB} & \boldsymbol{A^2B} & . & . & . & \boldsymbol{A^{n-1}B} \end{pmatrix} \begin{pmatrix} \beta_0(t) \\ \beta_1(t) \\ \beta_2(t) \\ . \\ . \\ \beta_{n-1}(t) \end{pmatrix}$$

- $x(t) = \boldsymbol{\phi\beta}(t)$

  $\boldsymbol{\beta}(t) = \boldsymbol{\phi}^{-1}\boldsymbol{x}(t)$

  $\boldsymbol{\phi}$ is non-singular and invertible

Controllability

Basic Pre-requisites

- Before we prove the condition for controllability let us see some basic prerequisites.
- Consider, the matrix

$$e^{\boldsymbol{A}t} = \boldsymbol{Q} \begin{bmatrix} e^{\lambda_1 t} & te^{\lambda_1 t} & t^2 e^{\lambda_1 t} & 0 & 0 \\ 0 & e^{\lambda_1 t} & te^{\lambda_1 t} & 0 & 0 \\ 0 & 0 & e^{\lambda_1 t} & 0 & 0 \\ 0 & 0 & 0 & e^{\lambda_1 t} & 0 \\ 0 & 0 & 0 & 0 & e^{\lambda_2 t} \end{bmatrix} \boldsymbol{Q}^{-1}$$

- We can see that every entry of $e^{\boldsymbol{A}t}$ is a linear combination of terms $[e^{\lambda_1 t}, te^{\lambda_1 t}, t^2 e^{\lambda_1 t}, e^{\lambda_2 t}]$.
- These values depend upon eigenvalues and their indices. Here,$\bar{n}_1 - 1 = 2$, where $\bar{n}_1$ is the index of the eigenvalue λ_1
- In general, if $\boldsymbol{A}$ has an eigenvalue with index $\bar{n}_1$, then every entry of $e^{\boldsymbol{A}t}$ is a linear combination of terms $e^{\lambda_1 t}, te^{\lambda_1 t}, t^2 e^{\lambda_1 t}, \cdots, t^{\bar{n}_1 - 1} e^{\lambda_1 t}$
- Every such term can be infinitely differentiable and can be expanded in a Taylor series at every 't' and is called Analytic.

Controllability

Basic Pre-requisites continued...

- Gram Matrix: Let A be the matrix whose columns are the vectors $\boldsymbol{v_1, v_2, \cdots, v_n}$. Then the Gram matrix is $\boldsymbol{A^T A}$, so $|\boldsymbol{G}| = |\boldsymbol{A}|^2$.
- It is the Hermitian matrix of the inner products whose entries are given by $\boldsymbol{G_{ij}} = \langle \boldsymbol{v_i, v_j} \rangle$
- In system theory, Controllability Gramian is used to determine whether or not a linear system is controllable.

Proof that Controllability Matrix has full row rank

- Now, let us prove that the controllability matrix has a full row rank.
- We can show that the controllability matrix $\mathcal{C}$ has a full row rank if $\boldsymbol{W_c}(t)$ is non singular, where, $\boldsymbol{W_c}(t) = \int_0^t e^{\boldsymbol{A}\tau} \boldsymbol{B}\boldsymbol{B}^T e^{\boldsymbol{A}^T \tau} d\tau$ is the controllability gramian matrix.

Controllability

- Consider the equation
  $\boldsymbol{x}(t_1) = e^{\boldsymbol{A}t}\boldsymbol{x}(0) + \int_0^t e^{\boldsymbol{A}(t-\tau)}\boldsymbol{B}u(\tau)d\tau$
- Let us claim that for any $\boldsymbol{x}(0) = x_0$ and any state $x(t_1) = x_1$, the input $u(t) = -\boldsymbol{B}^T e^{\boldsymbol{A}^T(t_1-t)}\boldsymbol{W}_c^{-1}(t_1)[e^{\boldsymbol{A}t}x_0 - x_1]$ will transfer the state x_0 to x_1 in the interval t_1
- Substituting $u(t)$ in the above equation we get ,

$$\begin{aligned} x(t_1) &= e^{\boldsymbol{A}t}\boldsymbol{x}(0) - \int_0^t e^{\boldsymbol{A}(t-\tau)}\boldsymbol{B}\boldsymbol{B}^T e^{\boldsymbol{A}^T(t_1-t)}\boldsymbol{W}_c^{-1}(t_1)[e^{\boldsymbol{A}t}x_0 - x_1]d\tau \\ x(1) &= e^{\boldsymbol{A}t}\boldsymbol{x}(0) - \boldsymbol{W}_c\boldsymbol{W}_c^{-1}(t_1)[e^{\boldsymbol{A}t}x_0 - x_1] \end{aligned}$$

- This shows that $(\boldsymbol{A}, \boldsymbol{B})$ is controllable if and only if $\boldsymbol{W}_c$ is invertible.

Controllability

- Suppose, $(\boldsymbol{A}, \boldsymbol{B})$ is controllable and $\boldsymbol{W}_c$ is a singular matrix, then there exists a non-zero $n \times 1$ vector $\boldsymbol{q}$ such that $\boldsymbol{q}^T \boldsymbol{W}_c \boldsymbol{q} = 0$

$$\begin{aligned}
\boldsymbol{q}^T \boldsymbol{W}_c &= \int_0^t \boldsymbol{q}^T e^{\boldsymbol{A}\tau} \boldsymbol{B}\boldsymbol{B}^T e^{\boldsymbol{A}^T\tau} \boldsymbol{q} d\tau \\
&= \int_0^t \left\| \boldsymbol{B}^T e^{\boldsymbol{A}^T\tau} \boldsymbol{q} \right\|^2 d\tau = 0 \\
&\Rightarrow \boldsymbol{B}^T e^{\boldsymbol{A}^T\tau} \boldsymbol{q} = \boldsymbol{q}^T e^{\boldsymbol{A}\tau} \boldsymbol{B} = 0
\end{aligned}$$

- Let,$\boldsymbol{\mathcal{C}} = e^{\boldsymbol{A}\tau} \boldsymbol{B}$
- We know that $e^{\boldsymbol{A}\tau} \boldsymbol{B}$ are analytic, i.e, every such term can be infinitely differentiable and can be expanded in a taylor series at every 't'.
- Therefore, $\boldsymbol{W}_c$ is non singular if and only if there exists no $n \times 1$ non zero vector $\boldsymbol{q}$ such that $\boldsymbol{q}^T \boldsymbol{\mathcal{C}} = 0$
- Therefore, $\boldsymbol{\mathcal{C}}$ has full row rank,i.e, rank of $\boldsymbol{\mathcal{C}} = n$

Controllability

Example-1 to check controllability

Consider the state and output equations,

$$\begin{aligned}\dot{x}_1 &= -2x_2 + u\\ \dot{x}_2 &= x_1 - 3x_2 + u\\ y &= x_1\end{aligned}$$

- Let us try to check for controllability of this system by transforming these equations into different forms.
- Representing these equations in matrix form , we get

$$\begin{aligned}\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix} &= \begin{bmatrix}0 & -2\\ 1 & -3\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} + \begin{bmatrix}1\\ 1\end{bmatrix}u\\ y &= \begin{bmatrix}1 & 0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}\end{aligned}$$

Controllability

Example-1 to check controllability

- We know that the controllability matrix for a system having two state variables is given by

$$\begin{aligned} \boldsymbol{\mathcal{C}} &= \begin{bmatrix} \boldsymbol{B} & \boldsymbol{AB} \end{bmatrix}_{2\times 2} \\ \therefore \boldsymbol{\mathcal{C}} &= \begin{bmatrix} 1 & -2 \\ 1 & -2 \end{bmatrix} \end{aligned}$$

- Rank of matrix $\boldsymbol{\mathcal{C}} = 1$ is less than 2. Therefore, $\boldsymbol{\mathcal{C}}$ is not a full row matrix.
- Therefore, the system is not Controllable.

Controllability

Example-1 to check controllability

- Now, let us check the controllability of the system by transforming it into diagonal canonical form.
- The transformation matrix $\boldsymbol{P}$ is the modal matrix formed by the eigenvectors corresponding to the eigenvalues of matrix $\boldsymbol{A}$.
- Matrix $\boldsymbol{P} = \begin{bmatrix} 1 & 1 \\ 1 & \frac{1}{2} \end{bmatrix}$
- Transformed matrices $\boldsymbol{A_z}$ and $\boldsymbol{B_z}$ are obtained using relations $\boldsymbol{A_z} = \boldsymbol{P^{-1}AP}, \boldsymbol{B_z} = \boldsymbol{P^{-1}B}$

$$\boldsymbol{A_z} = \begin{bmatrix} -2 & 0 \\ 0 & -1 \end{bmatrix} \boldsymbol{B_z} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Controllability

Example-1 to check controllability

- Representing them in matrix form , we get

$$\begin{bmatrix}\dot{z}_1\\ \dot{z}_2\end{bmatrix} = \begin{bmatrix}-2 & 0\\ 0 & -1\end{bmatrix}\begin{bmatrix}z_1\\ z_2\end{bmatrix} + \begin{bmatrix}1\\ 0\end{bmatrix}u$$

- Controllability matrix, $\boldsymbol{\mathcal{C}} = \begin{bmatrix}1 & -2\\ 0 & 0\end{bmatrix}$
- Rank of the matrix $\boldsymbol{\mathcal{C}} = 1$, has a row full of zeroes,is less than 2 .
- Therefore, the system is not Controllable.

Controllability

Example-1 to check controllability

- Now, let us check the controllability of the system by transforming it into Canonical Form-I using Similarity Transformation.
- The transformation matrix $\boldsymbol{P}$ is obtained by Conventional method, explained in Lecture-5.
- Matrix $\boldsymbol{P} = \begin{bmatrix} 1 & -1 \\ -1 & -2 \end{bmatrix}$
- Transformed matrices $\boldsymbol{A_z}$ and $\boldsymbol{B_z}$ are obtained using relations $\boldsymbol{A_z} = \boldsymbol{P^{-1}AP}, \boldsymbol{B_z} = \boldsymbol{P^{-1}B}$

$$\boldsymbol{A_z} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} \boldsymbol{B_z} = \begin{bmatrix} 1/3 \\ -2/3 \end{bmatrix}$$

Controllability

Example-1 to check controllability

- Representing them in matrix form , we get

$$\begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} + \begin{bmatrix} 1/3 \\ -2/3 \end{bmatrix} u$$

- Controllability matrix, $\boldsymbol{\mathcal{C}} = \begin{bmatrix} 1/3 & -2/3 \\ -2/3 & 4/3 \end{bmatrix}$
- Rank of the matrix $\boldsymbol{\mathcal{C}} = 1$, is less than 2 .
- Therefore, the system is not Controllable.

Controllability

Example-2 to check controllability

Consider the state and output equations,

$$\begin{aligned} \dot{x}_1 &= 3x_2 \\ \dot{x}_2 &= -2x_1 + 5x_2 + u \\ y &= x_1 \end{aligned}$$

- Let us try to check for controllability of this system by transforming these equations into different forms.
- Representing these equations in matrix form , we get

$$\begin{aligned} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} &= \begin{bmatrix} 0 & 3 \\ -2 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \\ y &= \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \end{aligned}$$

Controllability

Example-2 to check controllability

- We know that the controllability matrix for a system having two state variables is given by

$$\begin{aligned} \boldsymbol{\mathcal{C}} &= \begin{bmatrix} \boldsymbol{B} & \boldsymbol{AB} \end{bmatrix}_{2\times 2} \\ \therefore \boldsymbol{\mathcal{C}} &= \begin{bmatrix} 0 & 3 \\ 1 & 5 \end{bmatrix} \end{aligned}$$

- Rank of matrix $\boldsymbol{\mathcal{C}} = 1$ is 2. Therefore, $\boldsymbol{\mathcal{C}}$ is a full row matrix.
- Therefore, the system is Controllable.

Controllability

Example-2 to check controllability

- Now, let us check the controllability of the system by transforming it into diagonal canonical form.
- The transformation matrix $\boldsymbol{P}$ is the modal matrix formed by the eigenvectors corresponding to the eigenvalues of matrix $\boldsymbol{A}$.
- Matrix $\boldsymbol{P} = \begin{bmatrix} 1 & 1 \\ 2/3 & 1 \end{bmatrix}$
- Transformed matrices $\boldsymbol{A_z}$ and $\boldsymbol{B_z}$ are obtained using relations $\boldsymbol{A_z} = \boldsymbol{P^{-1}AP}, \boldsymbol{B_z} = \boldsymbol{P^{-1}B}$

$$\boldsymbol{A_z} = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} \boldsymbol{B_z} = \begin{bmatrix} 3 \\ -3 \end{bmatrix}$$

Controllability

Example-2 to check controllability

- Representing them in matrix form , we get

$$\begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} + \begin{bmatrix} 3 \\ -3 \end{bmatrix} u$$

- Controllability matrix, $\mathcal{C} = \begin{bmatrix} 3 & 6 \\ -3 & -9 \end{bmatrix}$
- Rank of the matrix $\mathcal{C} = 2$.
- Therefore, the system is Controllable.

Controllability

Example-2 to check controllability

- Now, let us check the controllability of the system by transforming it into Canonical Form-I using Similarity Transformation.
- The transformation matrix $\boldsymbol{P}$ is obtained by Conventional method, explained in Lecture-5.
- Matrix $\boldsymbol{P} = \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}$
- Transformed matrices $\boldsymbol{A_z}$ and $\boldsymbol{B_z}$ are obtained using relations $\boldsymbol{A_z} = \boldsymbol{P^{-1}AP}, \boldsymbol{B_z} = \boldsymbol{P^{-1}B}$

$$\boldsymbol{A_z} = \begin{bmatrix} 0 & 1 \\ -6 & 5 \end{bmatrix} \boldsymbol{B_z} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Controllability

Example-2 to check controllability

- Representing them in matrix form , we get

$$\begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -6 & 5 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

- Controllability matrix, $\boldsymbol{\mathcal{C}} = \begin{bmatrix} 0 & 1 \\ 1 & 5 \end{bmatrix}$
- Rank of the matrix $\boldsymbol{\mathcal{C}} = 2$.
- Therefore, the system is Controllable.