Control systems

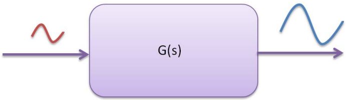
Frequency domain analysis

V. Sankaranarayanan

Outline

Outline

Frequency response



Definition

Steady state response of a linear system to a sinusoidal input

Advantages

- we can use the data obtained from measurements on the physical system without deriving its mathematical model.
- It is used to study both the absolute and relative stabilities of linear closed-loop systems from a knowledge of their open-loop frequency response characteristics.
- Frequency-response tests are, in general, simple and can be made accurately by use of readily available sinusoidal signal generators and precise measurement equipment.

Historical perspective

Development

- 1932 - Nyquist

Historical perspective

Development

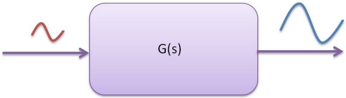
- 1932 - Nyquist
- 1940 - Frequency method - Bode

Historical perspective

Development

- 1932 - Nyquist
- 1940 - Frequency method - Bode
- 1950 - Zigler-Nichols methods for PID tuning

Frequency response analysis I



Definition

Steady state response of a linear system to a sinusoidal input

$$x(t) = X \sin \omega t$$

$$G(s) = \frac{p(s)}{q(s)} = \frac{p(s)}{(s+s_1)(s+s_2)....(s+s_n)}$$

$$Y(s) = G(s)X(s) = \frac{p(s)}{q(s)}X(s)$$

$$Y(s) = G(s)X(s) = G(s)\frac{\omega X}{s^2+\omega^2}$$

$$Y(s) = \frac{a}{s+j\omega} + \frac{\bar{a}}{s-j\omega} + \frac{b_1}{s+s_1} + \frac{b_2}{s+s_2} + ... + \frac{b_n}{s+s_n}$$

Frequency response analysis II

$$y(t) = ae^{-j\omega t} + \bar{a}e^{j\omega t} + b_1 e^{-s_1 t} + b_2 e^{-s_2 t} + \ldots\ldots + b_n e^{-s_n t}$$

$$y_{ss} = ae^{-j\omega t} + \bar{a}e^{j\omega t}$$

$$a = -\frac{XG(-j\omega)}{2j}$$

$$\bar{a} = \frac{XG(j\omega)}{2j}$$

Complex function $G(j\omega)$ can be written as

$$G(j\omega) = |G(j\omega)|e^{j\phi}$$

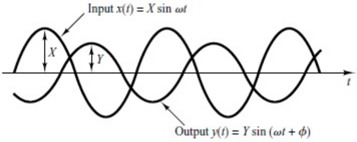
$$\phi = \angle G(j\omega) = \tan^{-1}\left[\frac{\text{imaginarypartof}G(j\omega)}{\text{realpartof}G(j\omega)}\right]$$

$$y_{ss}(t) = X|G(j\omega)|\frac{e^{j(\omega t+\phi)} - e^{-j(\omega t+\phi)}}{2j}$$

$$= X|G(j\omega)|\sin(\omega t + \phi)$$

$$= Y\sin(\omega t + \phi)$$

Frequency response analysis III



Outline

Example I

Consider the following system

$$G(s) = \frac{K}{1 + Ts}$$

$$G(j\omega) = \frac{K}{1 + Tj\omega}$$

$$|G(j\omega)| = \frac{K}{\sqrt{1 + T^2\omega^2}}$$

$$\phi = -\tan^{-1} T\omega$$

For the input $X \sin \omega t$

$$y_{ss} = \frac{KX}{\sqrt{1 + T^2\omega^2}} \sin(\omega t - \tan^{-1} T\omega)$$

Various methods

Different graphical techniques

- Bode diagram or logarithmic plot
- Nyquist plot or polar plot
- Log-magnitude-versus-phase plot (Nichols plots)

Outline

1. Introduction
   - Example

2. Graphical techniques
   - Bode plot
   - Examples
   - Polar plots

Bode plot

Bode plot

- It consists of two graphs
- Logarithm of the magnitude of a sinusoidal transfer function
- Phase angle
- Both are plotted against the frequency on a logarithmic
- logarithmic magnitude of $G(j\omega)$ is $20\log|G(j\omega)|$
- Log scale of frequency and the linear scale for magnitude or phase angle
- The multiplications of magnitude can be converted into addition
- Although it is not possible to plot the curves right down to zero frequency because of the logarithmic frequency ($\log 0 = \infty$), this does not create a serious problem

Bode plot

Basic factors

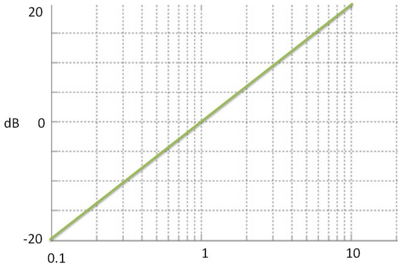
- Gain K
- Integral and derivative factors $(j\omega)^{\pm 1}$
- First-order factors $(1 + j\omega)^{\pm 1}$
- Quadratic factors $\left[1 + 2\zeta(j\frac{\omega}{\omega_n}) + (j\frac{\omega}{\omega_n})^2\right]^{\pm 1}$

Bode plot

Gain K

- A number greater than unity has a positive value in decibels, while a number smaller than unity has a negative value.
- The log-magnitude curve for a constant gain K is a horizontal straight line at the magnitude of $20 \log K$ decibels.
- The phase angle of the gain K is zero.
- The effect of varying the gain K in the transfer function is that it raises or lowers the log-magnitude curve of the transfer function

Gain K

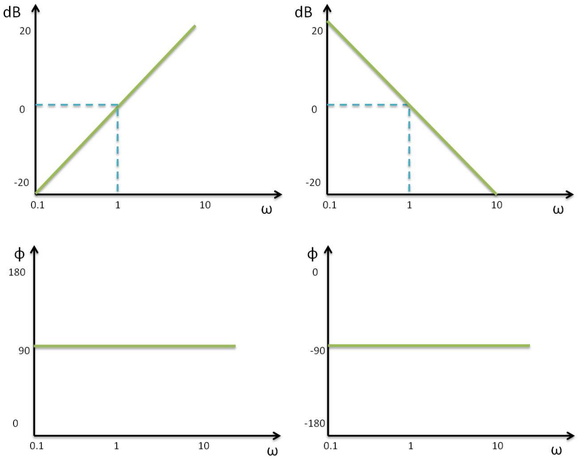


- $20\log(K * 10) = 20\log K + 20$
- $20\log(K \times 10^n) = 20\log K + 20n$
- $20\log K = -20\log \frac{1}{K}$

Integral and derivative factors

- $20\log\left|\frac{1}{j\omega}\right|$
- $-20\log\omega$dB
- The phase angle is -90^o
- The log magnitude is plotted against ω on the logarithmic scale
- $-20\log 1 = 0$dB and $-20\log 10 = -20$dB
- The slope is -20dB/decade
- For $20\log\omega$dB, the slope is 20dB/decade
- For $(\frac{1}{j\omega})^n$ or $(j\omega)^n$
- $20\log\left|\frac{1}{(j\omega)^n}\right| = -n \times 20\log|j\omega| = -20n\log\omega$
- The slope is $-20n$dB/decade or $20n$dB/decade

Integral and derivative factors



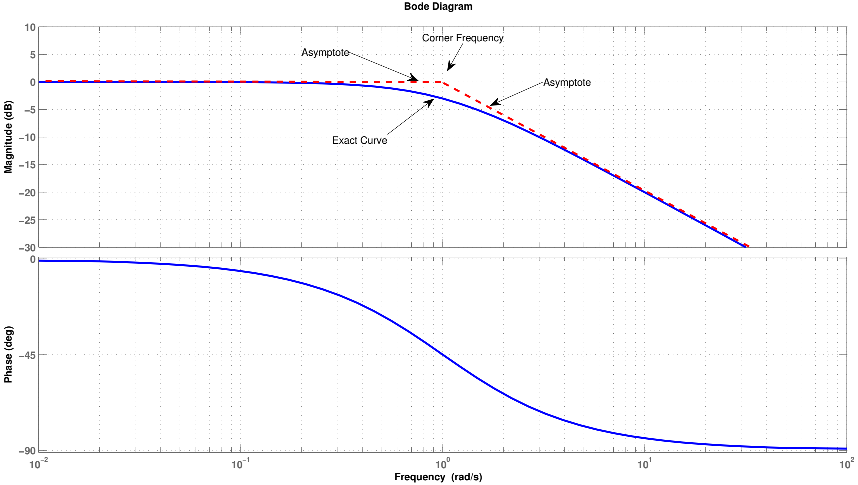
First order factors

- $20\log\left|\frac{1}{1+j\omega}\right|$
- $-20\log\sqrt{1+\omega^2T^2}\text{dB}$
- For low frequencies $\omega << 1/T$, it can be approximated to
- $-20\log\sqrt{1+\omega^2T^2} = -20\log 1 = 0\text{dB}$
- For high frequencies $\omega >> 1/T$, it can be approximated to
- $-20\log\sqrt{1+\omega^2T^2} = -20\log\omega T\text{dB}$
- At $\omega = 1/T$, the log magnitude equals 0 dB
- At $\omega = 10/T$, the log magnitude is -20 dB.
- For $\omega >> 1/T$, the log-magnitude curve is thus a straight line with a slope of -20 dB/decade

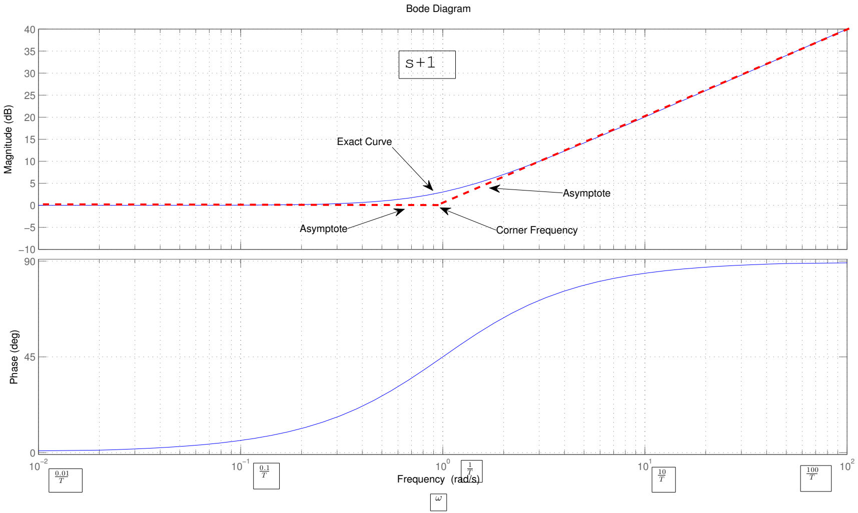
First order factors

- $\phi = -\tan^{-1}\omega T$
- At zero frequency $\phi = 0$
- At $\omega = 1/T$, $\phi = -45^o$
- At $\omega = \infty$, $\phi = -90^o$

First order factors



First order factors



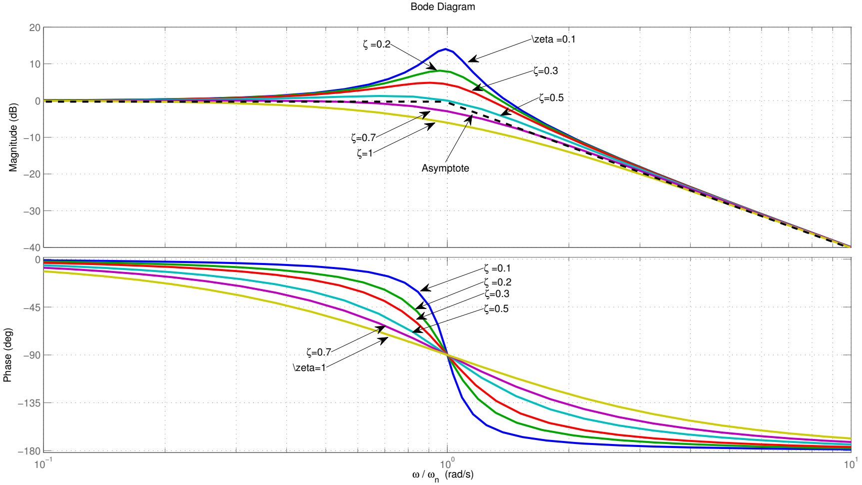
Second order factors

- $[1+2\zeta(j\frac{\omega}{\omega_n})+(j\frac{\omega}{\omega_n})^2]^{\pm 1}$
- $G(j\omega)=\frac{1}{1+2\zeta(j\frac{\omega}{\omega_n})+(j\frac{\omega}{\omega_n})^2}$
- $20\log\left|\frac{1}{1+2\zeta(j\frac{\omega}{\omega_n})+(j\frac{\omega}{\omega_n})^2}\right|=-20\log\sqrt{(1-\frac{\omega^2}{\omega_n^2})^2+(2\zeta\frac{\omega}{\omega_n})^2}$
- For low frequncies $\omega << \omega_n$
- $-20\log 1=0\text{dB}$
- For high frequencies
- $-20\log\frac{\omega^2}{\omega_n^2}=-40\log\frac{\omega}{\omega_n}\text{dB}$
- The slope is -40dB/decade
- At $\omega=\omega_n$
- $-40\log\frac{\omega_n}{\omega_n}=-40\log 1=0\text{dB}$

Second order factors

- $\phi = \angle \frac{1}{1+2\zeta(j\omega/\omega_n)+(j\omega/\omega_n)^2} = -\tan^{-1}\left[\frac{2\zeta\frac{\omega}{\omega_n}}{1-(\frac{\omega}{\omega_n})^2}\right]$
- At $\omega = 0$, $\phi = 0$
- At $\omega = \omega_n$, $\phi = -\tan^{-1}(\frac{2\zeta}{0}) = -\tan^{-1}\infty = -90^\circ$
- At $\omega = \infty$, $\phi = -180^\circ$

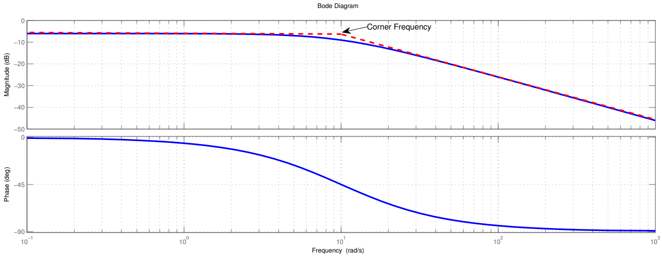
First order factors



Outline

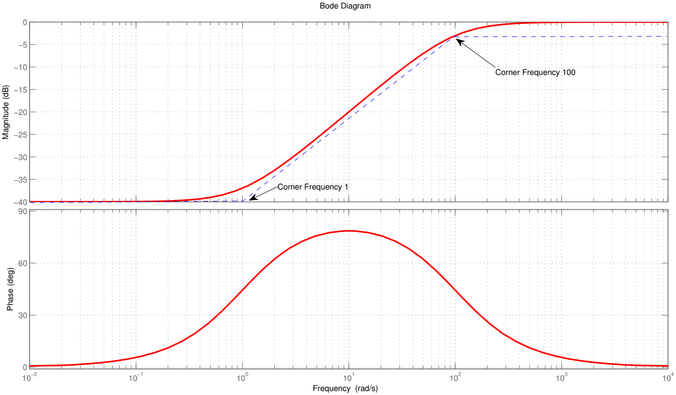
Examples

- $G(s) = \frac{5}{s+10}$
- Time constant form$G(s) = \frac{0.5}{1+0.1s}$
- DC Gain - $20 \log 0.5$
- First order factor - Till 10 rad/sec 0db/decade
- Slope -20dB/decade from 10 rad/sec

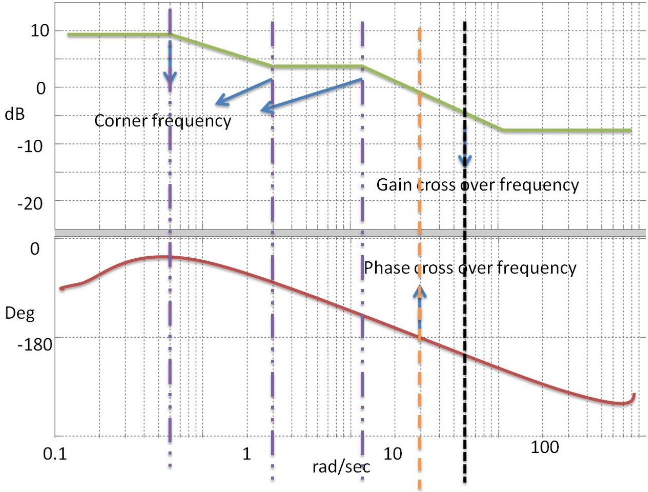


More Example

- $G(s) = \frac{s+1}{s+100}$ Time Constant Form $G(s) = \frac{0.01(s+1)}{0.01s+1}$
- Dc Gain -20log(0.01)=-40dB
- Slope : Before 1 rad/sec 0dB/decade ,1rad/sec-100 rad/sec 20dB/decade, After 100rad/sec 0db/decade



Terminology



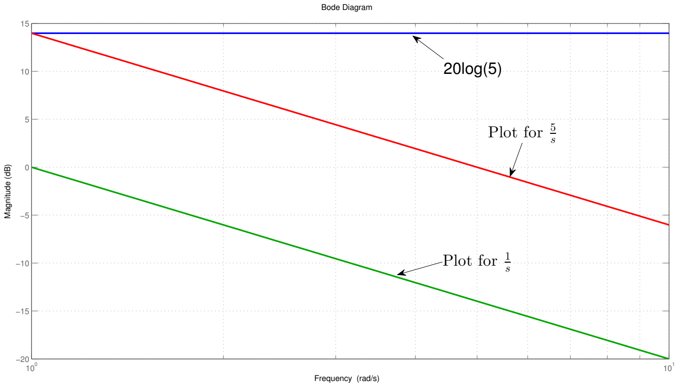
Magnitude Plot

Consider

$$G(s) = \frac{5}{s} = 5 \times \frac{1}{s}$$

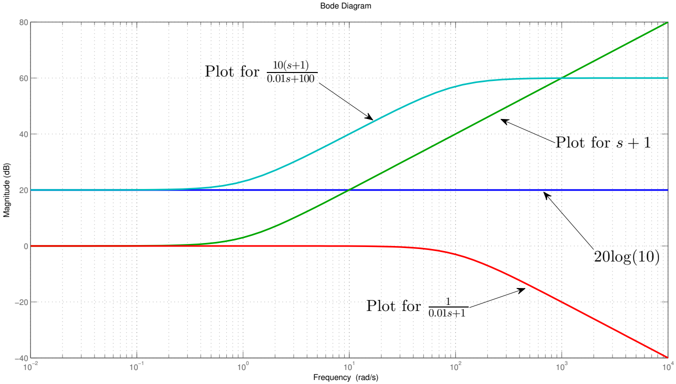
$$20log|G(s)| = 20log(5) + 20log|\frac{1}{s}|$$

$$BodeMagnitudePlot = ConstantLine + -20dBline$$



Magnitude Plot

$$G(s) = \frac{1000(s+1)}{s+100} = \frac{10(s+1)}{0.01s+1}$$

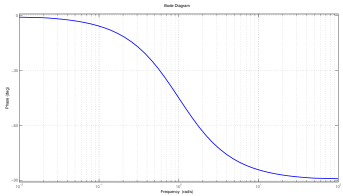


Phase Plot

Consider =

$$G(s) = \frac{1}{s+1}$$
$$G(j\omega) = \frac{1}{j\omega+1}$$

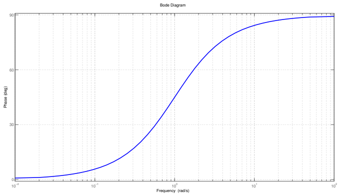
Angle is $\angle G(jw) = -\arctan(\omega)$
$\omega = 0, \angle G(j\omega) = 0^o$
$\omega = \infty, \angle G(j\omega) = -90^o$



Consider =

$$G(s) = s+1$$
$$G(j\omega) = j\omega+1$$

Angle is $\angle G(jw) = \arctan(\omega)$
$\omega = 0, \angle G(j\omega) = 0^o$
$\omega = \infty, \angle G(j\omega) = 90^o$



Sencond Order System

Consider

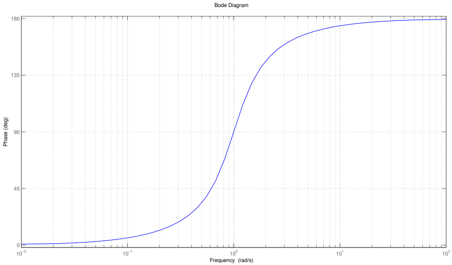
$$G(j\omega) = (j\omega)^2 + 2\zeta\omega_n j\omega + \omega_n^2$$

$$G(j\omega) = \omega_n^2 - \omega^2 + j2\zeta\omega_n\omega$$

Angle is

$$\angle G(j\omega) = \tan^{-1}\left(\frac{2\zeta\omega_n\omega}{\omega_n^2 - \omega^2}\right)$$

$\omega = 0, \angle G(j\omega) = 0^o$
$\omega = \infty, \angle G(j\omega) = 180^o$

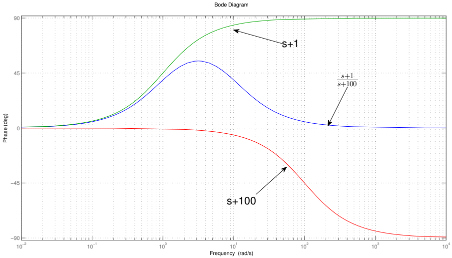


Example

$$G(s) = \frac{s+1}{s+100}$$

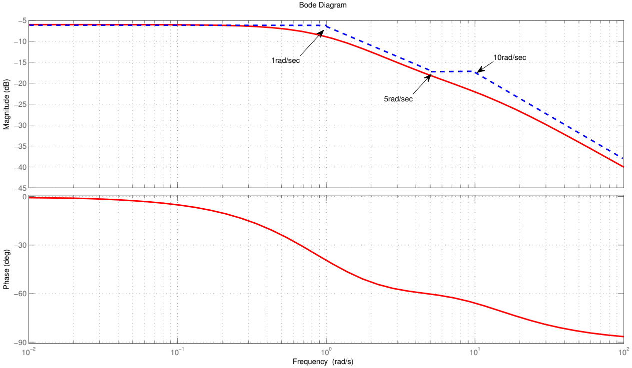
$$G(j\omega) = \frac{j\omega+1}{j\omega+10}$$

$\omega = 0 \angle G = 0, \omega = \infty \angle G = 0$ Supose $\omega = 30 rad/sec, \angle G = 16^o$



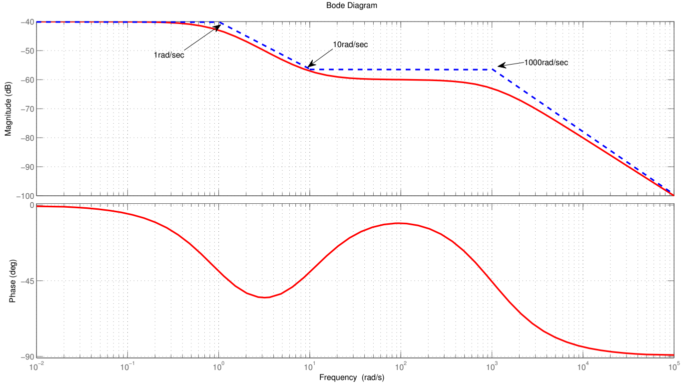
More Example

- $G(s) = \frac{s+5}{(s+1)(s+10)}$ Time Constant form $G(s) = 0.5\frac{0.2s+1}{(s+1)(0.1s+1)}$
- Dc Gain: -20log(0.5)
- Asymptote slopes: Upto 1rad/sec = 0dB/decade , 1rad/sec - 5rad/sec = -20dB/decade , 5rad/sec-10rad/sec = 0dB/decade, After 10rad/sec = -20dB/decade
- **Note:** Asymptotic approximation seems to fail because of presence of corner frequency near to each other.



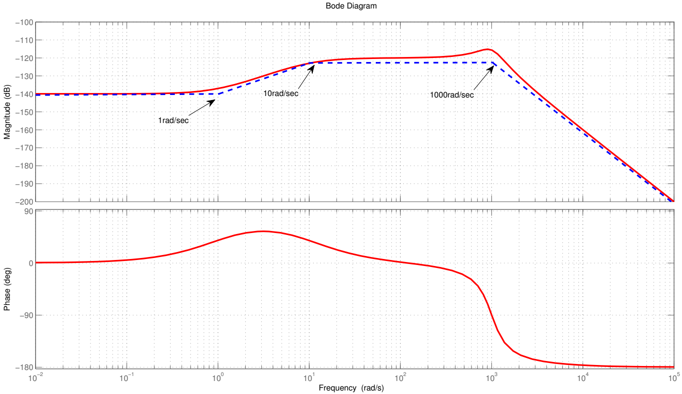
More Example

- $G(s) = \frac{s+10}{(s+1)(s+1000)}$ Time Constant form $G(s) = \frac{1}{100}\frac{0.1s+1}{(s+1)(0.001s+1)}$
- Dc Gain= -20log(0.01)
- Slope: Upto 1rad/sec = 0dB/decade , 1rad/sec - 10rad/sec = -20dB/decade , 10rad/sec - 1000rad/sec = 0dB/decade , After 1000rad/sec = -20dB/decade
- **Note:** Asymptotic approximation matches



More Example

- $G(s) = \dfrac{s+10}{(s+1)(s^2+600s+1000000)}$
- Slops: Upto 1rad/sec = 0dB/decade, 1rad/sec - 10rad/sec = 20dB/decade, 10rad/sec - 1000rad/sec= 0dB/decade, After 1000rad/sec= -40dB/decade



Phase margin and gain margin

Gain cross over frequency

It is the frequency at which $|G(j\omega)|$, the magnitude of the open-loop transfer function, is unity

Phase margin

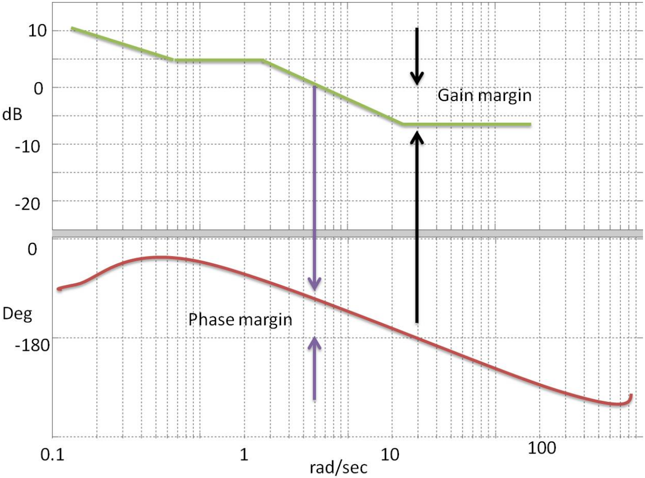
The phase margin is the amount of additional phase lag at the gain cross over frequency required to the verge of instability $\gamma = 180^o + \phi$

Gain margin

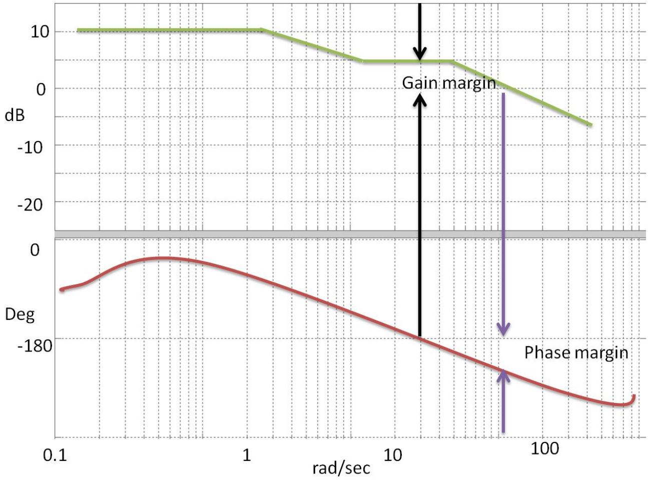
The gain margin is the reciprocal of the magnitude $|G(j\omega)|$ at the frequency at which the phase angle is -180^o(Phased cross over frequency)

$$K_g = \frac{1}{|G(j\omega)|} \quad = -20\log|G(j\omega)|$$

Terminology



Terminology

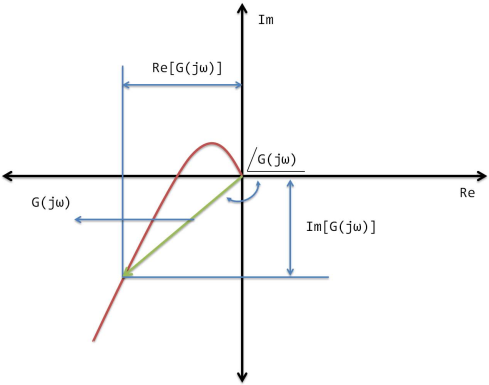


Outline

1. Introduction
   - Example
2. Graphical techniques
   - Bode plot
   - Examples
   - Polar plots

Polar plot

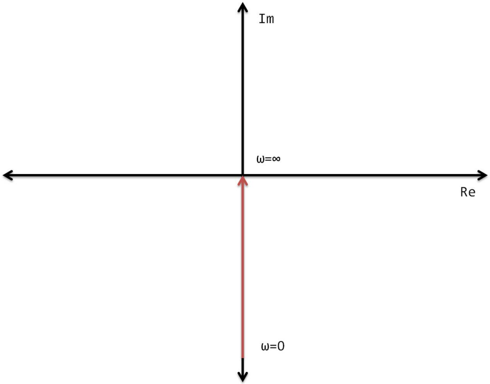
- It is a plot of the magnitude of $G(j\omega)$ versus the phase angle of $G(j\omega)$ on polar coordinates as ω is varied from zero to infinity
- In polar plots a positive (negative) phase angle is measured counterclockwise (clockwise) from the positive real axis



Integral and derivative factors

Integral and derivative factors

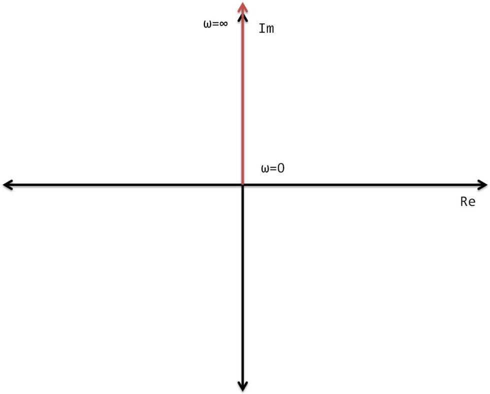
- $G(j\omega) = \frac{1}{j\omega} = -j\frac{1}{\omega} = \frac{1}{\omega}\angle -90^o$
- The Polar plot of $\frac{1}{j\omega}$ is the negative imaginary axis



Integral and derivative factors

Integral and derivative factors

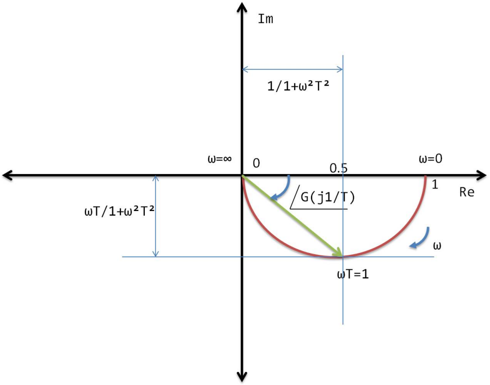
- The Polar plot of $j\omega$ is the positive imaginary axis



First order factors

First order factors

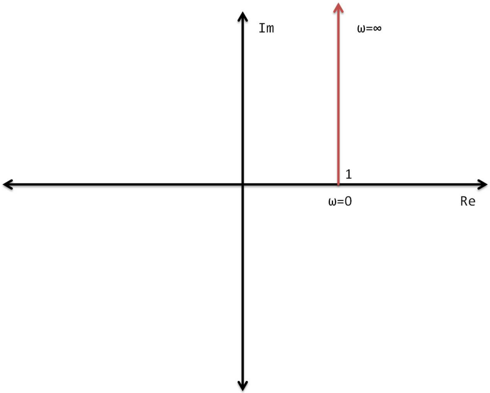
- $G(j\omega) = \frac{1}{1+j\omega T} = \frac{1}{\sqrt{1+\omega^2 T^2}} \angle -tan^{-1}\omega T$
- $G(j0) = 1\angle 0^o$, $G(j\frac{1}{T}) = \frac{1}{\sqrt{2}}\angle 45^o$
- $\omega \longrightarrow \infty$, $|G(j\omega)| \longrightarrow 0$ and $\angle G(j\omega) \longrightarrow -90^o$



First order factors

First order factors

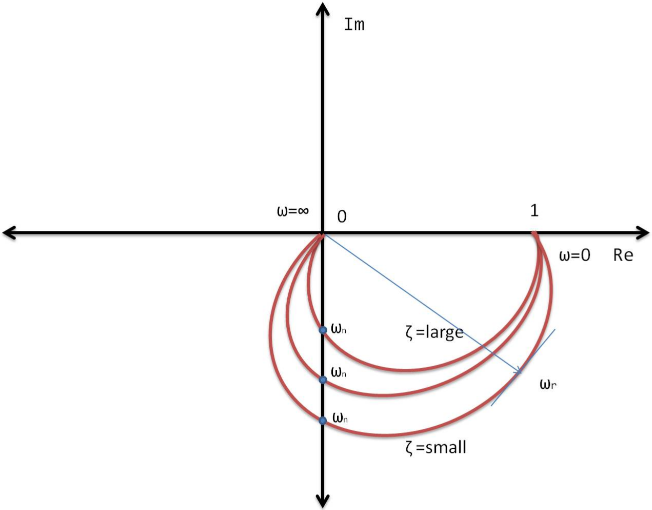
- $G(j\omega) = 1 + j\omega T$
- It is simply the upper half of the straight line passing through point $(1, 0)$ in the complex plane and parallel to the imaginary axis



Quadratic factors

Quadratic factors

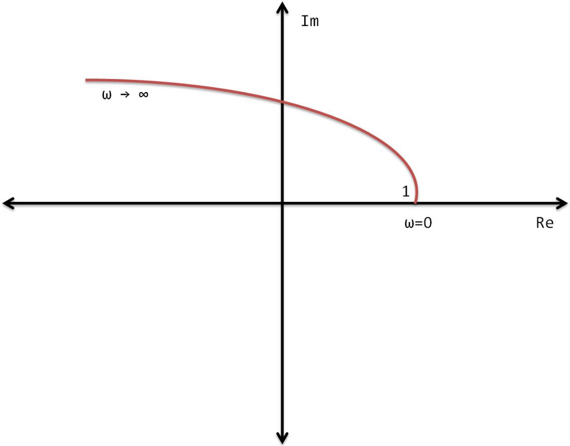
- $G(j\omega) = \frac{1}{1+2\zeta(j\frac{\omega}{\omega_n})+(j\frac{\omega}{\omega_n})^2}$
- For $\zeta > 0$
- $\lim_{\omega \to 0} G(j\omega) = 1\angle 0$
- $\lim_{\omega \to \infty} G(j\omega) = 0\angle -180$
- The polar plot of this sinusoidal transfer function starts at $1\angle 0$ and ends at $0\angle -180$ as ω increases from zero to infinity
- The high-frequency portion of $G(j\omega)$ is tangent to the negative real axis.
- For the underdamped case $\omega = \omega_n$, the phase angle is -90
- In the polar plot, the frequency point whose distance from the origin is maximum corresponds to the resonant frequency ω_r.


Im
ω=∞
0
1
ω=0
Re
ωn
ζ =large
ωn
ωr
ωn
ζ =small

Quadratic factors

Quadratic factors

- $G(j\omega) = 1 + 2\zeta(j\frac{\omega}{\omega_n}) + (j\frac{\omega}{\omega_n})^2$
- $= \left(1 - \frac{\omega^2}{\omega_n^2}\right) + j\left(\frac{2\zeta\omega}{\omega_n}\right)$
- $\lim_{\omega \to 0} G(j\omega) = 1\angle 0$
- $\lim_{\omega \to \infty} G(j\omega) = \infty\angle 180$


Im
ω → ∞
1
ω=0
Re